A dual descent algorithm for
node-capacitated multiflow problems and its applications

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October 5, 2015

Abstract

In this paper, we develop an $O((m \log k)\mathrm{MSF}(n, m, 1))$-time algorithm to find a half-integral node-capacitated multiflow of the maximum total flow-value in a network with n nodes, m edges, and k terminals, where $\mathrm{MSF}(n', m', \gamma)$ denotes the time complexity of solving the maximum submodular flow problem in a network with n' edges, m' edges, and the complexity γ of computing the exchange capacity of the submodular function describing the problem. By using Fujishige-Zhang algorithm for submodular flow, we can find a maximum half-integral multiflow in $O(mn^3 \log k)$ time. This is the first combinatorial strongly polynomial time algorithm for this problem. Our algorithm is designed on the basis of a developing theory of discrete convex functions on certain graph structures. Applications include "ellipsoid-free" combinatorial implementations of a 2-approximation algorithm for the minimum node-multiway cut problem by Garg, Vazirani, and Yannakakis.

Keywords: Node-capacitated multiflow, discrete convex analysis, submodular flow, node-multiway cut

1 Introduction

A node-capacitated undirected network is a quadruple $N = (V, E, S, c)$ of node set V, (undirected) edge set E, a specified subset S of nodes, called *terminals*, and a nonnegative integer-valued node capacity $c : V \setminus S \to \mathbf{Z}_+$ on nonterminal nodes. An *S-path* is a path connecting distinct terminals. A (node-capacitated) *multiflow* is a pair $(\mathcal{P}, \lambda)$ of a set $\mathcal{P}$ of S-paths and a flow-value function $\lambda : \mathcal{P} \to \mathbf{R}_+$ satisfying the node-capacity constraint:

$$\sum_{P \in \mathcal{P}:\ i \in V(P)} \lambda(P) \le c(i) \quad (i \in V \setminus S). \tag{1.1}$$

The total flow-value of a multiflow $f = (\mathcal{P}, \lambda)$ is defined as $\sum_{P \in \mathcal{P}} \lambda(P)$. A multiflow is called *maximum* if it has the maximum total flow-value among all possible multiflows. A multiflow $f = (\mathcal{P}, \lambda)$ is said to be *integral* if λ is integer-valued, and *half-integral* if 2λ is integer-valued.

In this paper, we address the problem of finding a maximum multiflow in a node-capacitated network. This multiflow problem appeared in the work by Garg, Vazirani, and Yannakakis [12] on an approximation algorithm for node-multiway cut. In fact, the LP-dual of our multiflow problem is a natural LP-relaxation of the *minimum node-multiway cut problem*; see also [27, Section 19.3]. They showed that this LP-dual always has a half-integral optimal solution. The half-integrality of the primal problem, i.e., the existence of a half-integral maximum multiflow,

was later shown by Pap [24, 25]. He also showed that a half-integral maximum multiflow can be found in strongly polynomial time.

In these works, the polynomial time solvability depends on the use of the ellipsoid method. Thus it is natural to seek a combinatorial polynomial time algorithm. For the case of unit node-capacity ($c(i) = 1$ for all $i \in V \setminus S$), Babenko [2] developed a combinatorial $O(mn^2)$ time algorithm to find a half-integral maximum multiflow, where n is the number of nodes and m is the number of edges. For general node-capacity, Babenko and Karzanov [3] developed a combinatorial weakly polynomial time algorithm to find a half-integral maximum multiflow. Their algorithm runs in $O(\mathrm{MF}(n, m, C)n^2 \log^2 n \log C)$ time, where $\mathrm{MF}(n, m, C)$ is the time complexity of solving the max-flow problem in a network with n nodes, m edges, and the maximum edge-capacity C.

The main result of this paper is the first combinatorial *strongly* polynomial time algorithm to solve the maximum node-capacitated multiflow problem. Our algorithm uses, as a sub-routine, an algorithm of solving the *maximum submodular flow problem*; see [9, Section 5.5 (c)]. Let $\mathrm{MSF}(n, m, \gamma)$ denote the time complexity of solving the maximum submodular flow problem on a network with n nodes, m edges, and the time complexity γ of computing the exchange capacity of the submodular function describing the problem.

Theorem 1.1. *There exists an $O((m \log k)\mathrm{MSF}(n, m, 1))$-time algorithm to find a half-integral maximum multiflow and a half-integral optimal dual solution in a network of n nodes, m edges, and k terminals.*

The current fastest maximum submodular flow algorithm is the push-relabel algorithm due to Fujishige and Zhang [11] of the time complexity $O(n^3\gamma)$; see the survey [10] on submodular flow algorithms. Thus we can solve the problem in $O(mn^3 \log k)$ time.

Application 1: Node-multiway cut. A *node-multiway cut* is a subset $X \subseteq V \setminus S$ of nonterminal nodes such that the deletion of X makes every pair of distinct terminals un-reachable, or equivalently, X meets every S-path. The *capacity* of a node-multiway cut X is defined as $\sum_{i \in X} c(i)$. The *minimum node-multiway cut problem* asks to find a node-multiway cut with the minimum capacity. This well-known NP-hard problem is naturally formulated as the following $\{0, 1\}$-integer program:

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{i \in V \setminus S} c(i)w(i) \\
\text{subject to} \quad & w : V \setminus S \to \{0, 1\}, \\
& \sum_{i \in V(P) \setminus S} w(i) \geq 1 \quad \text{(every S-path P).}
\end{aligned}
\tag{1.2}
$$

The natural LP-relaxation obtained by relaxing $w : V \setminus S \to \{0, 1\}$ into $w : V \setminus S \to \mathbf{R}_+$ is nothing but the LP-dual of our multiflow problem. As mentioned above, Garg, Vazirani, and Yannakakis [12] proved that a half-integral optimal LP solution $w^* : V \setminus S \to \{0, 1/2, 1\}$ always exists, and is obtained from any optimal LP solution by a simple rounding procedure; see [27, Section 19.3]. Then the set of nodes i with $w^*(i) \geq 1/2$ is a 2-approximation solution of the minimum node-multiway cut problem. This rounding algorithm needs an optimal LP solution, which is now obtained by our algorithm. To the best of our knowledge, this is the first combinatorial strongly polynomial time implementation of the 2-approximation algorithm.

Just recently, Chakuri and Madan [5] devised a simple method to round any feasible LP solution into a multiway cut of capacity within factor 2. Combining this rounding method with a fast FPTAS for multiflow (e.g., [13]), they obtain a considerably fast $(2 + \epsilon)$-approximation algorithm (with running time dependent of $1/\epsilon$).

2

Application 2: Integral multiflow. Our algorithm is also useful in the problem of finding a maximum integral multiflow. This problem is a capacitated version of openly-disjoint S-paths packing problem considered by Mader [21] (that corresponds to the case of $c(i) = 1$ ($i \in V \setminus S$)). Pap [24, 25] established the strongly polynomial time solvability of the maximum integral multiflow problem. The first step of his algorithm is to find a maximum half-integral multiflow. The second step is to construct and solve an instance of the openly-disjoint S-path packing problem (on the graph with size polynomial in the numbers of edges and nodes in the original network). Finally, combining the integer part of the half-integral multiflow with a solution of the packing problem, one obtains a maximum integer multiflow. The second step can be done by several combinatorial polynomial time algorithms, including [6] and [26, Section 73.1a]. Our algorithm can be used in the first step, and makes the whole algorithm fully combinatorial.

Outline. Let us outline our algorithm and the ideas behind it, as well as the structure of the paper. Our algorithm is designed on the basis of the following two ingredients. One is a combinatorial duality theory for a class of node-capacitated multiflow problems [15]. The other is a developing theory of discrete convex functions on certain graph structures [16, 17, 18], which aims to extend concepts in *Discrete Convex Analysis (DCA)* (Murota [22]) to tackle further various combinatorial optimization problems beyond network flows, matroids, and submodular functions. We will utilize these theories in a self-contained way.

In Section 3, following [15] we formulate the dual of our multiflow problem as a facility location problem on a tree. This formulation gives a fruitful combinatorial interpretation of the LP-dual problem (1.2), and brings a simple combinatorial algorithm to find a half-integral optimal multiflow from a given dual optimum, under a certain nondegeneracy assumption. We will deal with a perturbed problem satisfying this nondegeneracy assumption. Our goal is to solve this perturbed problem efficiently. We will see that the location problem is further formulated as an optimization over a certain discrete structure, and the objective is an *L-convex function on a Euclidean building* in the sense of [18]. This class of discrete convex functions shares many analogous properties with L-convex functions in DCA. In particular, as in the case of DCA, there is a natural descent algorithm, called the *steepest descent algorithm*, to minimize our L-convex function g. For each point x, the steepest descent algorithm chooses a point y (*steepest direction*) from a *discrete neighborhood* of x with smallest $g(y)$. If $g(y) = g(x)$, then x is guaranteed to be optimal. Otherwise, i.e., $g(y) < g(x)$, replace x by y, and repeat.

In Section 4, we will implement this conceptually simple algorithm. We will prove that in our case a steepest direction at each point can be found by solving one maximum submodular flow problem. This part is the heart of our analysis. As a consequence, we obtain an algorithm in a simple form as follows:

1. From a dual solution x, construct and solve an instance of the maximum submodular flow problem.

2. If the minimal minimum cut consists only of the source, then x is optimal, and an optimal multiflow is constructed from any maximum submodular flow. Otherwise the minimal minimum cut gives a steepest direction y of the neighborhood at x. Replace x by y, and go to 1.

Our maximum submodular flow problem is defined by a disjoint sum of submodular functions on 6-element sets. This enables us to compute the exchange capacity in constant time. Moreover, the number of iterations is estimated by the *geodesic descent property* (Theorem 3.5) of the steepest descent algorithm. This intriguing property says that a trajectory of the algorithm forms a geodesic to optimal solutions with respect to a certain l_∞-metric on the domain.

We know in advance the range where an optimum exists, and the diameter of the range is bounded by $O(m \log k)$ relative to the above metric. Consequently the number of iteration is bounded by $O(m \log k)$.

It should be noted that our algorithm design includes an interesting new technique of reducing bisubmodularity to submodularity. This technique and related arguments, including basics on submodularity, are summarized in Section 2. Actually step 1 of the above algorithm is essentially the feasibility check of a *bisubmodular flow* problem, that is, finding a (fractional) bidirected flow with the flow-boundary constrained to a bisubmodular polyhedron. This seemingly natural class of problems has not been well-studied so far. It is well-known that (fractional) bidirected flows are easily manipulated by ordinary flows in a *skew-symmetric network* obtained by doubling nodes and edges. We generalize this doubling construction to bisubmodular functions. We give a condition for a bisubmodular function f to be extended to a submodular function f' on a larger set, so that the bisubmodular polyhedron of f is a projection of the base polyhedron of f'. We show that a certain bisubmodular function on a 3-element set, which represents the flow-conservation and the node-capacity constraints on a node of degree 3, has such a submodular extension on a 6-element set. Our bisubmodular flow problem is described by the disjoint sum of these bisubmodular functions, and can be reduced to the submodular flow problem as mentioned above.

2 Preliminaries

Notation. Let $\mathbf{R}$, $\mathbf{R}_+$, $\mathbf{Z}$, and $\mathbf{Z}_+$ denote the sets of reals, nonnegative reals, integers, and nonnegative integers, respectively. The infinity element ∞ is treated as $x < \infty$ and $x + \infty = \infty$ for $x \in \mathbf{R}$. The set of all functions from a set V to a set R is denoted by R^V. For a function $g : V \to \mathbf{R} \cup \{\infty\}$, let $\operatorname{dom} g := \{x \in V \mid g(x) < \infty\}$. For a function $v \in \mathbf{R}^V$ and a subset $X \subseteq V$, let $v(X)$ denote $\sum_{x \in X} v(x)$. The function value $v(i)$ will also be denoted by v_i if no confusion occurs. For a (directed or undirected) graph $G = (V, E)$, an edge from i to j is denoted by ij. For a subset X of nodes, let δX denote the set of all edges leaving X. For an undirected graph Γ with a specified edge-length, let $d = d_\Gamma$ denote the shortest path metric on the vertex set with respect to the edge-length. In the following, graphs or networks are supposed to have no multiple edges and loops.

Signed set and transversal. A *signed set* U is the product $V \times \{+, -\}$ of a set V and the sign $\{+, -\}$. Elements $(i, +)$ and $(i, -)$ of U are simply denoted by i^+ and i^-, respectively. The *signed extension* of a set V is defined as the signed set $V \times \{+, -\}$ and is denoted by $V^\pm$. For $Y \subseteq V$, let $Y^+ := \{i^+ \mid i \in Y\}$ and $Y^- := \{i^- \mid i \in Y\}$. Also for $U \subseteq V^\pm$, let $U^+ := \{i^+ \mid i^+ \in U\}$ and $U^- := \{i^- \mid i^- \in U\}$. A subset X of $V^\pm$ is called *transversal* if $|X \cap \{i^+, i^-\}| \leq 1$ for all $i \in V$, and is called *co-transversal* if $|X \cap \{i^+, i^-\}| \geq 1$ for all $i \in V$. For $X \subseteq V^\pm$ let $\underline{X}$ denote the transversal obtained from X by deleting all $\{i^+, i^-\}$ with $\{i^+, i^-\} \subseteq X$, and let $\overline{X}$ denote the co-transversal obtained from X by adding all $\{i^+, i^-\}$ with $\{i^+, i^-\} \cap X = \emptyset$. For $u \in V^\pm$, define $\bar{u}$ by $\bar{u} := i^+$ if $u = i^-$ and $\bar{u} := i^-$ if $u = i^+$.

Skew-symmetric network. A *skew-symmetric network* (see e.g., [14]) is a directed network on a signed set such that an edge uv exists if and only if an edge $\bar{v}\bar{u}$ exists, and the (lower and upper) capacities of edges uv and $\bar{v}\bar{u}$ are the same. A skew-symmetric network is often useful for dealing with undirected objects.

2.1 Submodular flow

Here we summarize basics on submodular functions and submodular flows; see [8, 9, 10] for further details. A *submodular function* on a set V is a function ρ defined on 2^V satisfying

$\rho(X) + \rho(Y) \geq \rho(X \cap Y) + \rho(X \cup Y)$ for $X, Y \subseteq V$. Let ρ be a submodular function on V with $\rho(\emptyset) = 0$. The *base polyhedron* $\mathcal{B}(\rho)$ is defined to be the set of all vectors $x \in \mathbf{R}^V$ satisfying $x(X) \leq \rho(X)$ for $X \subseteq V$ and $x(V) = \rho(V)$. For $x \in \mathcal{B}(\rho)$ and a pair (i, j) of distinct elements of V, the *exchange capacity* $\kappa(x; i, j)$ at x is defined by

$$\kappa(x; i, j) := \max\{\alpha \in \mathbf{R}_+ \mid x + \alpha(\chi_i - \chi_j) \in \mathcal{B}(\rho)\},$$

where χ_i is the i-th unit vector defined by $\chi_i(j) := 1$ if $i = j$ and $\chi_i(j) := 0$ otherwise.

We next introduce submodular flows. Let N be a directed network with vertex set V, edge set A, edge-capacity $c : A \to \mathbf{R}_+$, and terminals $s, t \in V$. The set of nonterminal nodes is denoted by $U(:= V \setminus \{s, t\})$. We are given a submodular function $\rho : 2^U \to \mathbf{R}$ with $\rho(\emptyset) = \rho(U) = 0$. For a function $\varphi : A \to \mathbf{R}$, let $\nabla\varphi$ denote the function on V defined by

$$\nabla\varphi(i) := \sum\{\varphi(e) \mid e \in A\colon e \text{ enters } i\} - \sum\{\varphi(e) \mid e \in A\colon e \text{ leaves } i\}.$$

Namely $\nabla\varphi(i)$ represents the excess of φ at node i. An (s, t)-*flow*, or simply, a *flow* is a function $\varphi : A \to \mathbf{R}_+$ satisfying

$$0 \leq \varphi(e) \leq c(e) \quad (e \in A), \quad \nabla\varphi|_U \in \mathcal{B}(\rho),$$

where $(\cdot)|_U$ means the restriction to U. The flow-value of a flow φ is defined as $\nabla\varphi(t)(= -\nabla\varphi(s))$. An (s, t)-*cut* X is a subset of nodes containing s and not containing t. The *cut capacity* of X is defined as

$$c(\delta X) + \rho(X \setminus \{s\}). \tag{2.1}$$

An (s, t)-flow is called *maximum* if it has the maximum flow-value, and an (s, t)-cut is called *minimum* if it has the minimum capacity.

Theorem 2.1 (see [9, Theorem 5.11]). *The maximum flow-value of an (s, t)-flow is equal to the minimum cut-capacity of an (s, t)-cut X. If c and ρ are both integer-valued, then there exists an integer-valued maximum flow. For any maximum flow φ, the set of nodes reachable from s in the residual network N_φ of φ is the unique minimal minimum (s, t)-cut.*

Here the *residual network* N_φ of φ is a directed network on V constructed as follows: For each edge ij in N with $\varphi(ij) < c(ij)$, add an edge ij to N_φ. For each edge ij in N with $0 < \varphi(ij)$, add an edge ji to N_φ. For each pair of distinct nodes i, j in U with $\kappa(\nabla\varphi|_U; i, j) > 0$, add an edge ij to N_φ.

There are several combinatorial polynomial time algorithms for computing an integral maximum submodular flow, under the assumption that an oracle of computing the exchange capacity is available. They are designed by extending existing max-flow algorithms; see the survey [10] for further detail.

We note one basic property for the case where the network is skew-symmetric.

Lemma 2.2. *Suppose that V is a signed set with $t = \bar{s}$, and N is skew-symmetric. If ρ satisfies $\rho(\underline{X}) \leq \rho(X)$ for all $X \subseteq U$, then the unique minimal minimum (s, t)-cut is a transversal.*

Proof. By the assumption for ρ, it suffices to show $c(\delta\underline{X}) \leq c(\delta X)$. Suppose that some edge $uv(\not\in \delta X)$ appears in $\delta\underline{X}$. In this case, it holds $u \in \underline{X} \subseteq X$ and $v \in X \setminus \underline{X}$. This means that $\{v, \bar{v}\} \subseteq X$ and $\bar{u} \not\in X$. Hence edge $\bar{v}\bar{u} \in \delta X$ (of the same capacity) does not appear in $\delta\underline{X}$. Consequently the cut-capacity does not increase. $\square$

2.2   Submodular extension

Let $V := \{1, 2, \dots, n\}$. Let 3^V denote the set of ordered pairs of disjoint subsets of V. For a function h on 3^V, the polyhedron $\mathcal{D}(h)$ in $\mathbf{R}^V$ is defined to be the set of all vectors $z \in \mathbf{R}^V$ satisfying

$$z(Y) - z(Z) \le h(Y, Z) \quad ((Y, Z) \in 3^V). \tag{2.2}$$

If h is a bisubmodular function[1], then $\mathcal{D}(h)$ is known as a bisubmodular polyhedron. We study the case where $\mathcal{D}(h)$ is a projection of the base polyhedron of some submodular function. A representative of such polyhedra is the polyhedron of all flow-boundaries of a bidirected network; see [1].

Let $V^\pm = \{1^+, 2^+, \dots, n^+, 1^-, 2^-, \dots, n^-\}$ be the signed extension of V. For a function h on 3^V, a *normal submodular extension* of h is a submodular function ρ on $2^{V^\pm}$ with $\rho(\emptyset) = 0$ satisfying

$$\rho(Y^+ \cup Z^-) = h(Y, Z) \quad ((Y, Z) \in 3^V), \tag{2.3}$$

$$\rho(\overline{X}) = \rho(\underline{X}) \le \rho(X) \quad (X \in 2^{V^\pm}). \tag{2.4}$$

Define a map (projection) $\phi : \mathbf{R}^{V^\pm} \to \mathbf{R}^V$ by

$$(\phi(x))(i) := \frac{x(i^+) - x(i^-)}{2} \quad (x \in \mathbf{R}^{V^\pm}, i \in V). \tag{2.5}$$

Lemma 2.3. *Let h be a function on 3^V and ρ a normal submodular extension of h. Then it holds $\phi(\mathcal{B}(\rho)) = \mathcal{D}(h)$.*

Proof. Note that $\rho(V^\pm) = \rho(\emptyset) = 0$. We first show that $\phi(\mathcal{B}(\rho)) \subseteq \mathcal{D}(h)$. Take an arbitrary x in $\mathcal{B}(\rho)$. For $(Y, Z) \in 3^V$, let $X := Y^+ \cup Z^-$. Note that $X = \underline{X}$ and $\rho(X) = \rho(\underline{X}) = \rho(\overline{X})$. Then we have

$$x(X) \le \rho(X), \ x(\overline{X}) \le \rho(\overline{X}) = \rho(X), \ x(V^\pm) = \rho(V^\pm) = 0.$$

Hence we have $(x(X) + x(\overline{X}) - x(V^\pm))/2 \le (\rho(X) + \rho(\overline{X}) - \rho(V^\pm))/2 = \rho(X) = h(Y, Z)$. Also we have

$$
\begin{aligned}
(x(X) + x(\overline{X}) - x(V^\pm))/2 &= x(Y^+) + x(Z^-) - (x(Y^+ \cup Z^+) + x(Y^- \cup Z^-))/2 \\
&= (x(Y^+) - x(Y^-))/2 - (x(Z^+) - x(Z^-))/2 \\
&= (\phi(x))(Y) - (\phi(x))(Z).
\end{aligned}
$$

Hence $\phi(x)$ belongs to $\mathcal{D}(h)$.

Next we show the converse. Take an arbitrary $z \in \mathcal{D}(h)$. Define a vector x in $\mathbf{R}^{V^\pm}$ by $x(i^+) := z(i)$ and $x(i^-) := -z(i)$ for $i \in V$. Obviously $\phi(x) = z$. It suffices to show that x belongs to $\mathcal{B}(\rho)$. Since $x(V^\pm) = 0$, we have $x(V^\pm) = 0 = \rho(\emptyset) = \rho(V^\pm)$. For $X \subseteq V^\pm$, we have

$$x(X) = x(\underline{X}) = \sum_{i : i^+ \in \underline{X}} z(i) - \sum_{i : i^- \in \underline{X}} z(i) \le h(\underline{X}^+, \underline{X}^-) = \rho(\underline{X}) \le \rho(X).$$

Thus $x \in \mathcal{B}(\rho)$, and hence $\mathcal{D}(h) \subseteq \phi(\mathcal{B}(\rho))$. $\qquad\square$

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[1] A function h on 3^V is called *bisubmodular* if it satisfies

$$h(X, Y) + h(X', Y') \ge h(X \cap X', Y \cap Y') + h((X \cup X') \setminus (Y \cup Y'), (Y \cup Y') \setminus (X \cup X')) \quad ((X, Y), (X', Y') \in 3^V).$$

If $h : 3^V \to \mathbf{R}$ has a normal submodular extension, then h is necessarily a bisubmodular function. Not all bisubmodular functions admit submodular extensions (Y. Iwamasa 2015).

We consider a special bisubmodular function on 3-element set $\{1, 2, 3\}$, which plays a key role in Section 3. For $b \geq 0$, let Δ_b be the function on $3^{\{1,2,3\}}$ defined by

$$\Delta_b(Y, Z) := \begin{cases} 2b & \text{if } |Y| \geq 2, \\ b & \text{if } |Y| = 1, |Z| \leq 1, \\ 0 & \text{otherwise } (Y = \emptyset \text{ or } |Z| \geq 2), \end{cases} \qquad (Y, Z) \in 3^V. \qquad (2.6)$$

Lemma 2.4. *The polyhedron $\mathcal{D}(\Delta_b)$ is the set of nonnegative vectors $z \in \mathbf{R}_+^{\{1,2,3\}}$ satisfying*

$$z(1) + z(2) + z(3) \leq 2b,$$
$$z(1) - z(2) - z(3) \leq 0,$$
$$-z(1) + z(2) - z(3) \leq 0,$$
$$-z(1) - z(2) + z(3) \leq 0.$$

Proof. Observe that these inequalities appears in (2.2). So it suffices to show that inequalities (2.2) are implied by the above inequalities. This is a routine verification. For example, $z(1) + z(2) - z(3) \leq \Delta_b(\{1, 2\}, \{3\}) = 2b$ is obtained by adding $z(1) + z(2) + z(3) \leq 2b$ and $-z(3) \leq 0$. Also $z(1) \leq \Delta_b(\{1\}) = b$ is implied by $z(1) + z(2) + z(3) \leq 2b$ and $z(1) - z(2) - z(3) \leq 0$, and $z(1) - z(2) \leq \Delta_b(\{1\}, \{2\}) = b$ is implied by $z(1) - z(2) - z(3) \leq 0$ and $z(3) \leq b$. $\qquad \square$

The polyhedron $\mathcal{D}(\Delta_b)$ is a simplex of vertices $(0, 0, 0)$, $(b, b, 0)$, $(b, 0, b)$ and $(0, b, b)$. We will see in Section 3 that $\mathcal{D}(\Delta_b)$ represents the flow-conservation law and the node-capacity constraint on a node of degree 3. This bisubmodular function Δ_b has a normal submodular extension. The following example was found by Yuni Iwamasa via a computer calculation. Notice that each $X \subseteq \{1^+, 2^+, 3^+, 1^-, 2^-, 3^-\}$ is one of the following six types:

type 1: $|X^+| \geq 2$ and $|X^-| \leq 1$.

type 2: $X^+ = \{i^+\}$ and $X^- = \{1^-, 2^-, 3^-\} \setminus \{i^-\}$ for some $i \in \{1, 2, 3\}$.

type 3: $X \subseteq \{1^-, 2^-, 3^-\}$ or $\{1^-, 2^-, 3^-\} \subseteq X$.

type 4: $|X^+| = 2$ and $|X^-| = 2$.

type 5: $X^+ = \{i^+\}$, $|X^-| \leq 2$, and $X^- \neq \{1^-, 2^-, 3^-\} \setminus \{i^-\}$ for some $i \in \{1, 2, 3\}$.

type 6: $X = \{1^+, 2^+, 3^+, 1^-, 2^-, 3^-\} \setminus \{i^-\}$ for some $i \in \{1, 2, 3\}$.

Define $\Delta_b^* : 2^{\{1^+, 2^+, 3^+, 1^-, 2^-, 3^-\}} \to \mathbf{R}$ by

$$\Delta_b^*(X) := \begin{cases} 2b & \text{if } X: \text{type 1}, \\ 0 & \text{if } X: \text{type 2 or 3}, \\ b & \text{otherwise } (X: \text{type 4, 5, or 6}). \end{cases} \qquad (2.7)$$

Lemma 2.5. *Δ_b^* is a normal submodular extension of Δ_b.*

Proof. It suffices to consider the case of $b = 1$; we denote Δ_1 and Δ_1^* by Δ and Δ^*, respectively.

First we show (2.3). For $(Y, Z) \in 3^{\{1,2,3\}}$, let $X := Y^+ \cup Z^-$. Then $X^+ = Y^+$ and $X^- = Z^-$. If $|Y| = |X^+| \geq 2$, then $|Z| = |X^-| \leq 1$, and X is of type 1; hence $\Delta^*(X) = 2 = \Delta(Y, Z)$. If $|Y| = |X^+| = 1$ and $|Z| = |X^-| \leq 1$. Then X is of type 5, and hence $\Delta^*(X) = 1 = \Delta(Y, Z)$. If Y is empty, then $X \subseteq \{1^-, 2^-, 3^-\}$, and X is type 3; hence

7

$\Delta^*(X) = 0 = \Delta(Y, Z)$. If $|Z| = |X^-| \geq 2$ and $Y \neq \emptyset$, then $|Y| = |X^+| = 1$, and X is of type 2; hence $\Delta^*(X) = 0 = \Delta(Y, Z)$.

Second we show (2.4). It suffices to show that $\Delta^*(X) = \Delta^*(\overline{X})$ holds for any transversal X, and that $\Delta^*(\underline{X}) \leq \Delta^*(X)$ holds for any X that is not transversal and not co-transversal. The former property follows from $\Delta^*(\{1^+\}) = 1 = \Delta^*(\{1^+, 2^+, 3^+, 2^-, 3^-\})$, $\Delta^*(\{1^-\}) = 0 = \Delta^*(\{2^+, 3^+, 1^-, 2^-, 3^-\})$, $\Delta^*(\{1^+, 2^+\}) = 2 = \Delta^*(\{1^+, 2^+, 3^+, 3^-\})$, $\Delta^*(\{1^+, 2^-\}) = 1 = \Delta^*(\{1^+, 3^+, 2^-, 3^-\})$, and $\Delta^*(\{1^-, 2^-\}) = 0 = \Delta^*(\{3^+, 1^-, 2^-, 3^-\})$. The latter property follows from $\Delta^*(\{2^+, 2^-\}) = 1 > 0 = \Delta^*(\emptyset)$, $\Delta^*(\{1^+, 2^+, 2^-\}) = 2 > 1 = \Delta^*(\{1^+\})$, and $\Delta^*(\{2^+, 1^-, 2^-\}) = 1 > 0 = \Delta^*(\{1^-\})$.

Finally we show the submodularity of Δ^*. Take $X, Y \subseteq \{1^+, 2^+, 3^+, 1^-, 2^-, 3^-\}$. We can assume that $X \not\subseteq Y$ and $Y \not\subseteq X$.

Case 1: X is of type 6. In this case, $X \cup Y$ is the whole set, and is of type 3. Therefore it suffices to consider the case where $X \cap Y$ is of type 1 and Y is not of type 1. Necessarily Y is of type 4 or 6. Thus submodular inequality $1 + 1 \geq 2 + 0$ holds.

Case 2: X is of type 2 with $1^+ \in X$. If Y is also of type 2, then $X \cup Y (\supseteq \{1^-, 2^-, 3^-\})$ is of type 3, $X \cap Y (\subseteq \{1^-, 2^-, 3^-\})$ is also of type 3. If Y is of type 3 with $Y \subseteq \{1^-, 2^-, 3^-\}$, then both $X \cap Y$ and $X \cup Y$ are of type 3. If Y is of type 3 with $Y \supseteq \{1^-, 2^-, 3^-\}$, then Y cannot have 1^+, and thus both $X \cap Y$ and $X \cup Y$ are of type 3. In these cases, submodularity $(0 + 0 \geq 0 + 0)$ holds. Thus we may assume that Y is of type 1,4, or 5. Observe that neither $X \cap Y$ nor $X \cup Y$ is of type 1. We may assume that Y contains 1^+ and does not contain 1^-. otherwise $X \cap Y$ or $X \cup Y$ is of type 3, and submodularity holds. Necessarily Y is of type 1. Then $X \cap Y$ is of type 5, and $X \cup Y$ is of type 4 or 6; submodularity $(0 + 2 \geq 1 + 1)$ holds.

Case 3: X is of type 3 and Y is not of type 2. If Y is also of type 3, then both $X \cap Y$ and $X \cup Y$ is of type 3; submodularity holds. Since one of $X \cap Y$ and $X \cup Y$ is of type 3, it suffices to consider the case where $X \cap Y$ or $X \cup Y$ is of type 1. We show that Y is also type 1. If $X \cap Y$ is of type 1, then $X = \{1^+, 2^+, 3^+, 1^-, 2^-, 3^-\} \setminus \{i^+\}$, and necessarily Y is of type 1. If $X \cup Y$ is of type 1, then $X = \{i^-\}$, and Y consists of at least two elements in $\{1^+, 2^+, 3^+\}$ (type 1).

Case 4: X is of type 4 or 5, and Y is type 1, 4, or 5. Suppose that X is of type 4. Then $X \cup Y$ is not type 1. We may consider the case where $X \cap Y$ is of type 1 and Y is not of type 1. Then Y contains X^+. Thus Y is of type 4, and $X \cup Y$ necessarily contains $\{1^-, 2^-, 3^-\}$ (type 3); submodularity $(1 + 1 \geq 2 + 0)$ holds. Suppose that X is of type 5 with $1^+ \in X$. Then $X \cap Y$ is not type 1. We may consider the case where $X \cup Y$ is of type 1. If Y does not contain 1^+, then $X \cap Y$ is of type 3; submodularity $(1 + 1 \geq 2 + 0)$ holds. Thus $|Y^+| \geq 2$ and $|Y^-| = 0$ or 1; Y is of type 1. The intersection $X \cap Y$ is of type 5; thus $1 + 2 \geq 2 + 1$ holds. $\square$

3 Node-capacitated multiflow

In this section, we introduce a combinatorial duality theory, developed by [15], for a class of node-capacitated multiflow problems. We consider the following multiflow problem. Now assume that network N also has a nonnegative edge-cost $a : E \to \mathbf{R}_+$; so the network is a 5-tuple (V, E, S, c, a). For a multiflow $f = (\mathcal{P}, \lambda)$, the total flow-values on node i and edge e are denoted by $f(i) := \sum_{P \in \mathcal{P}:\ i \in V(P)} \lambda(P)$ and $f(e) := \sum_{P \in \mathcal{P}:\ e \in E(P)} \lambda(P)$, respectively. The *cost* $a(f)$ is defined by

$$a(f) := \sum_{e \in E} a(e)f(e).$$

Next we define the value of a multiflow. A *tree-embedding* $\mathcal{E} = (\Gamma, \{q_s\}_{s \in S})$ is a pair of a tree Γ and a family $\{q_s\}_{s \in S}$ of vertices of Γ indexed by terminal set S. The *$\mathcal{E}$-value* $v_{\mathcal{E}}(f)$ of a

8

multiflow $f = (\mathcal{P}, \lambda)$ is defined by

$$v_{\mathcal{E}}(f) := \sum_{P \in \mathcal{P}} d(q_{s_P}, q_{t_P}) \lambda(P),$$

where s_P, t_P denote the ends of an S-path P, and $d = d_\Gamma$ denotes the shortest path metric of Γ with respect to unit edge-length. We are now ready to define our multiflow problem. An instance of the problem is a pair of a network $N = (V, E, S, c, a)$ and a tree-embedding $\mathcal{E} = (\Gamma, \{p_s\}_{s \in S})$, and the task is to find a multiflow f that maximizes $v_{\mathcal{E}}(f) - a(f)$.

This somewhat artificial formulation turns out to be useful, and actually generalizes the original problem. Indeed, take Γ as a star with $|S|$ leaves v_s ($s \in S$), let $\mathcal{E} := (\Gamma, \{v_s\}_{s \in S})$, and let $a(e) := 0$ for each edge e. Then $v_{\mathcal{E}}(f) - a(f)$ is twice the total flow-value of f.

In Section 3.1, we present a combinatorial duality theorem and an optimality criterion. We introduce a nondegeneracy concept of the problem, and give an algorithm to find a half-integral optimal multiflow from a dual optimum under the nondegeneracy assumption. We also explain how to reduce the original problem to a nondegenerate problem. In Section 3.2, we see that our dual objective can be viewed as an L-convex function on a certain graph structure, and present the steepest descent algorithm (SDA) to minimize L-convex functions and its iteration bound.

3.1 Duality

Let a pair of $N = (V, E, S, c, a)$ and $\mathcal{E} = (\Gamma, \{q_s\}_{s \in S})$ be an instance of the problem. The vertex set of Γ is also denoted by Γ (instead of $V(\Gamma)$). Let Γ^* denote the edge-subdivision of Γ, where $\Gamma \subseteq \Gamma^*$, the edge-length of Γ^* is defined as $1/2$ uniformly, and the shortest path metric d_{Γ^*} is also denoted by d.

A pair (p, r) of a tree-valued function $p : V \to \Gamma^*$ and a nonnegative half-integer-valued function $r : V \to \mathbf{Z}_+/2$ is called a *potential* if it satisfies the following conditions:

(p1) For each node i, $r(i)$ is an integer if and only if $p(i)$ belongs to Γ.

(p2) For each edge ij, it holds $d(p(i), p(j)) - r(i) - r(j) \leq a(ij)$.

(p3) For each terminal s, it holds $(p(s), r(s)) = (q_s, 0)$.

Then the following min-max formula and optimality criterion hold:

Theorem 3.1 ([15])**.** *Suppose that a is even-integer-valued. The maximum of $v_{\mathcal{E}}(f) - a(f)$ over all multiflows f is equal to the minimum of $\sum_{i \in V \setminus S} 2c(i)r(i)$ over all potentials (p, r).*

Lemma 3.2 ([15])**.** *Suppose that a is even-integer-valued. A multiflow $f = (\mathcal{P}, \lambda)$ and a potential (p, r) are both optimal if and only if they satisfy the following conditions:*

(o1) *For each path P in $\mathcal{P}$ with $\lambda(P) > 0$, it holds $d(q_{s_P}, q_{t_P}) = \sum_{ij \in E(P)} d(p(i), p(j))$.*

(o2) *For each edge ij with $f(ij) > 0$, it holds $d(p(i), p(j)) - r(i) - r(j) = a(ij)$.*

(o3) *For each nonterminal node i with $r(i) > 0$, it holds $f(i) = c(i)$.*

We will use the if part (and the weak duality in Theorem 3.1) only, which is proved for completeness.

Proof. (If part). For any multiflow $f = (\mathcal{P}, \lambda)$ and any potential (p, r), the difference $\sum_{i \in V \setminus S} 2c_i r_i - (v_{\mathcal{E}}(f) - a(f))$ is equal to

$$\sum_{i \in V \setminus S} 2(c_i - f_i) r_i + \sum_{ij \in E} f_{ij}(a_{ij} - d(p_i, p_j) + r_i + r_j)$$

$$+ \sum_{P \in \mathcal{P}} \lambda(P) \left(\sum_{ij \in E(P)} d(p_i, p_j) - d(q_{s_P}, q_{t_P}) \right) \geq 0, \tag{3.1}$$

where we use

$$\sum_{ij \in E} f_{ij} d(p_i, p_j) = \sum_{P \in \mathcal{P}} \lambda(P) \sum_{ij \in E(P)} d(p_i, p_j),$$

$$\sum_{i \in V \setminus S} 2 f_i r_i = \sum_{ij \in E} f_{ij}(r_i + r_j).$$

Thus, if f and (p, r) satisfy conditions (o1), (o2), and (o3), then the equality holds in (3.1), and both f and (p, r) are optimal. □

Nondegenerate case. An instance $(N, \mathcal{E})$ is said to be *nondegenerate* if the edge-costs a are positive even-valued and the degree of each node in Γ is at most 3. Suppose that $(N, \mathcal{E})$ is nondegenerate. We further assume, for notational simplicity, that tree Γ has no vertex of degree one (by attaching paths of infinite length). Let Γ_2 and Γ_3 denote the sets of vertices of Γ with degree 2 and 3, respectively. For a vertex v in Γ, the neighbors of v in Γ are denoted by $v_{\to 1}, v_{\to 2}$ if $v \in \Gamma_2$ and $v_{\to 1}, v_{\to 2}, v_{\to 3}$ if $v \in \Gamma_3$. Let Γ^* denote the edge-subdivision of Γ. For vertex $v \in \Gamma^*$, the neighbors of v in Γ^* are denoted by $v_{\to *1}, v_{\to *2}$ if $v \in \Gamma_2$ or $v \in \Gamma^* \setminus \Gamma$, and $v_{\to *1}, v_{\to *2}, v_{\to *3}$ if $v \in \Gamma_3$. Let $\Gamma^*_{v,k}$ denote the connected component of $\Gamma^* - v$ containing $v_{\to *k}$.

We are going to characterize the flow support of an optimal multiflow. Let (p, r) be a potential. Motivated by (o2), define the edge subset $E_{p,r}$ by

$$E_{p,r} := \{ ij \in E \mid d(p(i), p(j)) - r(i) - r(j) = a_{ij} \}.$$

For a nonterminal node i, let $\delta_{p,k}(i)$ denote the set of edges $ij \in E_{p,r}$ with $p(j) \in \Gamma^*_{p(i),k}$. Since each edge cost $a(ij)$ is positive, it holds $p(i) \neq p(j)$ for $ij \in E_{p,r}$. Thus $\delta_{p,k}(i)$ for $k = 1, 2, 3$ (or $k = 1, 2$) partition the set $\delta\{i\}$ of all edges in $E_{p,r}$ incident to i.

A (p, r)-*admissible support* is a function $\zeta : E_{p,r} \to \mathbf{R}_+$ satisfying the following conditions:

(a1) For each nonterminal node i with $p(i) \notin \Gamma_3$, it holds $\zeta(\delta_{p,1}(i)) = \zeta(\delta_{p,2}(i)) \leq c(i)$.

(a2) For each nonterminal node i with $p(i) \in \Gamma_3$, it holds

$$\zeta(\delta_{p,1}(i)) + \zeta(\delta_{p,2}(i)) + \zeta(\delta_{p,3}(i)) \leq 2c(i),$$
$$\zeta(\delta_{p,1}(i)) - \zeta(\delta_{p,2}(i)) - \zeta(\delta_{p,3}(i)) \leq 0,$$
$$-\zeta(\delta_{p,1}(i)) + \zeta(\delta_{p,2}(i)) - \zeta(\delta_{p,3}(i)) \leq 0,$$
$$-\zeta(\delta_{p,1}(i)) - \zeta(\delta_{p,2}(i)) + \zeta(\delta_{p,3}(i)) \leq 0.$$

(a3) For each nonterminal node i with $r(i) > 0$, it holds $\zeta(\delta\{i\}) = 2c(i)$.

(a4) For each edge e, $\zeta(e)$ is a half-integer, and for each nonterminal node i, $\zeta(\delta\{i\})$ is an integer.

10

Recall the notational convention $\zeta(\delta_{p,k}(i)) := \sum_{e \in \delta_{p,k}(i)} \zeta(e)$. Notice that a (p,r)-admissible support is viewed as an edge-weight whose degree vector belongs to a bisubmodular polyhedron described by $\Delta_{c(i)}$.

It is not difficult to see from Lemma 3.2 that for any half-integral optimal multiflow f, the flow-support ζ of f, defined by $\zeta(e) := f(e)$, is a (p,r)-admissible support. The converse also holds.

Lemma 3.3 ([15]). *Let (p,r) be a potential. If a (p,r)-admissible support ζ exists and is given, then (p,r) is optimal and a half-integral optimal multiflow is obtained in $O(nm)$ time.*

Thus our problem is to find a potential (p,r) so that a (p,r)-admissible support exists. An algorithm for Lemma 3.3 is the following.

Algorithm 1: Construction of an optimal multiflow from a (p,r)-admissible support.

Input: A potential (p,r) and a (p,r)-admissible support ζ.

Output: A half-integral optimal multiflow $f = (\mathcal{P}, \lambda)$.

Step 0: $\mathcal{P} = \emptyset$.

Step 1: Choose a terminal s and an edge sj with $\zeta(sj) > 0$. If such a terminal does not exist, then $f = (\mathcal{P}, \lambda)$ is a half-integral optimal multiflow; stop. Otherwise let $j_0 \leftarrow s$, $j_1 \leftarrow j$, $\mu \leftarrow \zeta(sj)$, $l \leftarrow 1$, and go to step 2.

Step 2: If j_l is a terminal, then add path $P = (j_0, j_1, \ldots, j_l)$ to $\mathcal{P}$ with flow-value $\lambda(P) := \mu$, let $\zeta(e) \leftarrow \zeta(e) - \mu$ for each edge e in P, and go to step 1. Otherwise go to step 3.

Step 3: If $p(j_l) \notin \Gamma_3$ and $j_{l-1}j_l \in \delta_{p,k}(j_l)$ for $k \in \{1,2\}$, then choose an edge $j_l j_{l+1}$ from $\delta_{p,k'}(j_l)$ with $k' \neq k$ and $\zeta(j_l j_{l+1}) > 0$, and let $\mu \leftarrow \min\{\mu, \zeta(j_l j_{l+1})\}$.

If $p(j_l) \in \Gamma_3$ and $j_{l-1}j_l \in \delta_{p,k}(j_l)$ for $k \in \{1,2,3\}$, then choose an edge $j_{l+1}j_l$ from $\delta_{p,k'}(j_l)$ with $k' \neq k$, $\zeta(j_l j_{l+1}) > 0$, and $\zeta(\delta_{p,k}(j_l)) + \zeta(\delta_{p,k'}(j_l)) - \zeta(\delta_{p,k''}(j_l)) > 0$ for $k'' \in \{1,2,3\} \setminus \{k,k'\}$. Let

$$\mu \leftarrow \min\left\{\mu, \zeta(j_l j_{l+1}), \frac{\zeta(\delta_{p,k}(j_l)) + \zeta(\delta_{p,k'}(j_l)) - \zeta(\delta_{p,k''}(j_l))}{2}\right\}.$$

Let $l \leftarrow l + 1$ and go to step 2.

This algorithm is essentially the proof of [15, Lemma 4.5]. Let us sketch the correctness of the algorithm; we show that the resulting multiflow f satisfies the conditions (o1),(o2), and (o3) in Lemma 3.2 with (p,r). The condition (o2) follows from $f(e) = 0$ for $e \in E \setminus E_{p,r}$. In step 3, we can always choose a required edge by (a1) and (a2). Also ζ still satisfies the conditions (a1), (a2), and (a4), thanks to the way of the update. Each produced path $(j_0, j_1, j_2, \ldots, j_m)$ satisfies

$$d(p(j_{l-1}), p(j_l)) + d(p(j_l), p(j_{l+1})) = d(p(j_{l-1}), p(j_{l+1})) \quad (1 \leq l \leq m-1)$$

since $p(j_{l-1}) \in \Gamma^*_{p(j_l),k}$ and $p(j_{l+1}) \in \Gamma^*_{p(j_l),k'}$ for $k \neq k'$. Also each $d(p(j_{l-1}), p(j_l))$ is positive (since a is positive). Since Γ is a tree, we have $d(p(j_0), p(j_m)) = \sum_{l=1}^{m} d(p(j_{l-1}), p(j_l))$; see e.g., [17, Lemma 3.9]. Thus each produced path satisfies (o1). By the same argument, every edge e with $\zeta(e) > 0$ extends to an S-path consisting of edges e' with $\zeta(e') > 0$ satisfying (o1). This means that if no terminal s is chosen in step 1, then $\zeta = 0$. By (a3), the resulting multiflow f satisfies (o3). Notice that μ is a half-integer by (a4). Hence f is half-integral and optimal. Once an S-path P is obtained, ζ becomes zero on some edge, or

$\zeta(\delta_{p,k}(i)) + \zeta(\delta_{p,k'}(i)) - \zeta(\delta_{p,k''}(i))$ becomes zero on some node i; they remain zero in subsequent iterations. Thus the algorithm terminates after $O(m)$ paths are obtained, where each path is found in $O(n)$ time by keeping $\{e \in \delta_{p,k}(i) \mid \zeta(e) > 0\}$ ($i \in V, k = 1, 2, 3$) as lists.

We estimate the range in which an optimal potential exists. Let Γ_0 denote the minimal subtree in Γ containing $\{q_s\}_{s \in S}$, and let $d(\Gamma_0)$ denote the diameter of Γ_0, i.e., $d(\Gamma_0) := \max_{u,v \in \Gamma_0} d(u, v)$.

Lemma 3.4. *There is an optimal potential (p, r) with $p(i) \in \Gamma_0$ and $r(i) \leq d(\Gamma_0)$ for $i \in V$.*

Proof. Let (p, r) be a potential. Suppose that there is a nonterminal node i^* with $p_{i^*} \notin \Gamma_0$. Then there is $s \in S$ such that the path between q_s and p_{i^*} is outside of Γ_0, Take such s, i^* with $d(q_s, p_{i^*})$ maximum. We can assume that $\Gamma_{p_{i^*},1}^*$ contains q_s. Let X be the set of nodes j with $p_j = p_{i^*}$. Suppose that $p_{i^*} \in \Gamma^* \setminus \Gamma$. Then $r_j \geq 1/2$ for all $j \in X$. For each $j \in X$, replace (p_j, r_j) by $(p_{j \to^* 1}, r_j - 1/2)$. For an edge ij with $i \in X$ and $j \notin X$, both $d(p_i, p_j)$ and $r_i + r_j$ decrease by $1/2$, and thus (p2) remains to hold. For other edge ij, quantity $d(p_i, p_j) - r_i - r_j$ is nonincreasing or remains nonpositive (if $i, j \in X$). The feasibility (p2) still holds (since $a(ij)$ is nonnegative). Thus the resulting (p, r) is a potential, and the objective value decreases. Suppose that $p_{i^*} \in \Gamma$. For each $j \in X$, replace (p_j, r_j) by $(p_{j \to 1}, r_j)$. For each edge ij, distance $d(p_i, p_j)$ does not increase. Thus the feasibility (p2) holds, and the objective value does not change. By repeating this procedure, we can make (p, r) so that $p_i \in \Gamma_0$ for $i \in V$, without increasing the objective value. Suppose that $r_i > d(\Gamma_0)$ for some i; necessarily $r_i \geq 1$. For each edge ij connecting i, $d(p_i, p_j) - r_i - r_j - a_{ij} \leq -1$ (since $d(p_i, p_j) \leq d(\Gamma_0)$ and $d(p_i, p_j) - r_i - r_j$ is an integer). Thus we can replace r_i by $r_i - 1$ to decrease the objective value. Repeating this procedure, (p, r) satisfies $r_i \leq d(\Gamma_0)$, as required. $\square$

Reduction to a nondegenerate instance. Here we explain how to reduce our original problem to a nondegenerate problem. An instance of the original problem is viewed as a pair of network $N = (V, E, S, c, a)$ and a tree-embedding $\mathcal{E} = (\Gamma, \{v_s\}_{s \in S})$ such that $a(e) = 0$ for all edges e and Γ is a star with center v_0 and leaves v_s ($s \in S$). We are going to construct a nondegenerate instance. Define edge cost $\tilde{a}$ by $\tilde{a}(e) := 2$ for each edge $e \in E$. Let $\tilde{N} := (V, E, S, c, \tilde{a})$. Next we define a tree-embedding $\tilde{\mathcal{E}} = (\tilde{\Gamma}, \{q_s\}_{s \in S})$. Let Σ be any (finite) trivalent tree with $|S|$ leaves u_s ($s \in S$) and diameter $D = O(\log |S|)$. For each $s \in S$, consider an infinite path P_s having a vertex u_s' of degree one. Identify u_s and u_s', i.e., glue P_s and Σ at u_s. The resulting infinite tree is denoted by $\tilde{\Gamma}$. Define q_s as the vertex in P_s having distance $(2|E| + 1)D$ from $u_s(= u_s')$. See Figure 1 for the construction of $\tilde{\Gamma}$.

Now we obtain a nondegenerate instance $(\tilde{N}, \tilde{\mathcal{E}})$. Let $(\tilde{p}, \tilde{r})$ and $f = (\mathcal{P}, \lambda)$ be an optimal potential and an optimal multiflow, respectively, for this perturbed instance $(\tilde{N}, \tilde{\mathcal{E}})$. We show that f is a maximum multiflow, i.e., optimal for the original instance $(N, \mathcal{E})$. We are going to construct an optimal potential (p, r) for $(N, \mathcal{E})$ from $(\tilde{p}, \tilde{r})$. Let B_i denote the subgraph of $\tilde{\Gamma}$ induced by vertices q with $d(\tilde{p}_i, q) \leq \tilde{r}_i$. Namely B_i is (the intersection of Γ and) the ball with center $\tilde{p}_i$ and radius $\tilde{r}_i$. Then it holds

$$d(B_i, B_j) := \min_{u \in B_i, v \in B_j} d(u, v) = \min\{0, d(\tilde{p}_i, \tilde{p}_j) - \tilde{r}_i - \tilde{r}_j\}. \tag{3.2}$$

For $k = 0, 1, 2, \ldots, 2|E|$ and $s \in S$, let $F_{s,k}$ denote the subgraph of $\tilde{\Gamma}$ consisting of edges uv in $P_s \subseteq \tilde{\Gamma}$ such that $kD \leq d(u_s, u) = d(u_s, v) - 1 < (k + 1)D$. Let F_k be the union of $F_{s,k}$ over $s \in S$. We say that an edge ij in E *hits* F_k if $B_i \cap B_j = \emptyset$ and the path between B_i and B_j meets an edge of F_k. For an edge ij, it holds $d(B_i, B_j) \leq a_{ij} \leq 2$. Therefore the path between B_i and B_j consists of at most two edges. Since $2|E| + 1$ subgraphs $F_0, F_1, \ldots, F_{2|E|}$ are edge-disjoint, there is an index k such that every edge in E does not hit F_k. Fix such an index k.
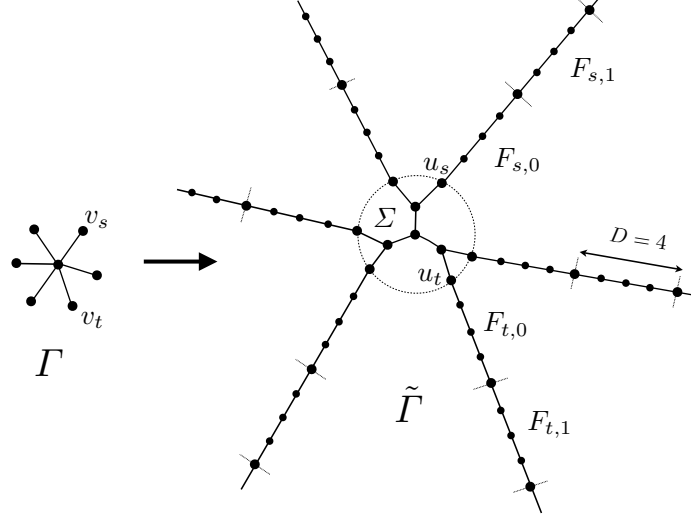
Figure 1: Construction of $\tilde{\Gamma}$

For each $s \in S$, choose the edge e_s of $F_{s,k}$ nearest to q_s (furthest from u_s). Delete all e_s from $\tilde{\Gamma}$. There are $|S| + 1$ connected components C_0, C_s ($s \in S$), where C_0 is the connected component containing Σ, and C_s is the connected component containing q_s. For each $i \in V$, define $(p_i, r_i) \in \Gamma^* \boxtimes \mathbf{Z}^*$ by

$$(p_i, r_i) := \begin{cases} (v_0, 1) & \text{if } B_i \text{ contains two } e_s, e_{s'}, \\ (\bar{v}_s, 1/2) & \text{if } B_i \text{ contains exactly one } e_s, \\ (v_s, 0) & \text{if } B_i \text{ is contained in } C_s \text{ for } s \in S \cup \{0\}, \end{cases} \tag{3.3}$$

where $\bar{v}_s$ denotes the vertex in $\Gamma^* \setminus \Gamma$ obtained by subdividing edge $v_0 v_s$ in Γ. We show that (p, r) is a potential for $(N, \mathcal{E})$ and satisfies (o1), (o2), and (o3) with f. We first show the feasibility (p2) $d_\Gamma(p_i, p_j) \leq r_i + r_j$ for $ij \in E$. Since $p_i \in \Gamma^* \setminus \Gamma$ implies $r_i = 1/2$, we may consider the three cases: (i) $p_i = v_s$, $p_j = v_0$, (ii) $p_i = v_s$, $p_j = \bar{v}_{s'}$ for $s \neq s'$, and (iii) $p_i = v_s$, $p_j = v_{s'}$ for $s \neq s'$. For (i), B_j necessarily contains two $e_s, e_{s'}$; otherwise ij hits F_k at e_s. This implies $r_j = 1$. (ii) and (iii) cannot occur since, otherwise, ij hits F_k at e_s. Thus (p2) holds, and hence (p, r) is a potential; (p1) and (p3) are clearly satisfied.

Next we show (o1),(o2), and (o3) for f and (p, r). To show (o2), take an edge ij with $f(ij) > 0$. By (o2) for f and $(\tilde{p}, \tilde{r})$ in $(\tilde{N}, \tilde{\mathcal{E}})$, it holds $d_{\tilde{\Gamma}}(\tilde{p}_i, \tilde{p}_j) - \tilde{r}_i - \tilde{r}_j = 2$. Thus the balls B_i and B_j are disjoint and have distance 2. Suppose that B_i has two edges $e_s, e_{s'}$. Then B_i must meet $F_{s,k}$ for every $s \in S$. Necessarily B_j cannot be contained by C_0. Also B_j cannot have e_t for any $t \in S \setminus \{s, s'\}$; otherwise ij hits F_k. Thus B_j is contained in C_t for $t \in S$, and $(p_i, r_i) = (v_0, 1)$ and $(p_j, r_j) = (v_t, 0)$ hold. If B_i has (only one) e_s and B_j has (only one) $e_{s'}$, then s and s' must be different, and necessarily $(p_i, r_i) = (\bar{v}_s, 1/2)$ and $(p_j, r_j) = (\bar{v}_{s'}, 1/2)$. If B_i has only e_s and B_j does not have any of e_t, then necessarily B_j is contained in C_s or C_0; hence $(p_i, r_i) = (\bar{v}_s, 1/2)$ and $(p_j, r_j) = (v_s, 0)$ or $(v_0, 0)$. If both B_i and B_j do not have any of e_s, then both B_i and B_j are contained in C_s for some $s \in S \cup \{0\}$, and $p_i = p_j$ and $r_i = r_j = 0$. In all the cases, it holds $d_\Gamma(p_i, p_j) = r_i + r_j$, implying (o2).

Consider the condition (o1). Take a path $P = (s = j_0, j_1, \ldots, j_m = t)$ with $\lambda(P) > 0$. There is an index l such that B_l contains e_s; otherwise F_k is hit by some edge. Moreover such an index l is unique. Otherwise, the balls B_{j_l} and $B_{j_{l'}}$ with $l < l'$ contain e_s. Then $d(\tilde{p}_{j_l}, \tilde{p}_{j_{l'}}) - \tilde{r}_{j_l} - \tilde{r}_{j_{l'}} < 0$. However, by (o1) and (o2) for f and $(\tilde{p}, \tilde{r})$, we have $d(\tilde{p}_{j_l}, \tilde{p}_{j_{l'}}) = \sum_{i=l}^{l'-1} d(\tilde{p}_{j_i}, \tilde{p}_{j_{i+1}}) = \sum_{i=l}^{l'-1} \tilde{r}_{j_i} + \tilde{r}_{j_{i+1}} + 2$, and $d(\tilde{p}_{j_l}, \tilde{p}_{j_{l'}}) - \tilde{r}_{j_l} - \tilde{r}_{j_{l'}} \geq 2 > 0$; a contradiction. Similarly there is a unique index l' such that $B_{l'}$ contains e_t. If $l = l'$, then $(p_{j_0}, p_{j_1}, \ldots, p_{j_m})$
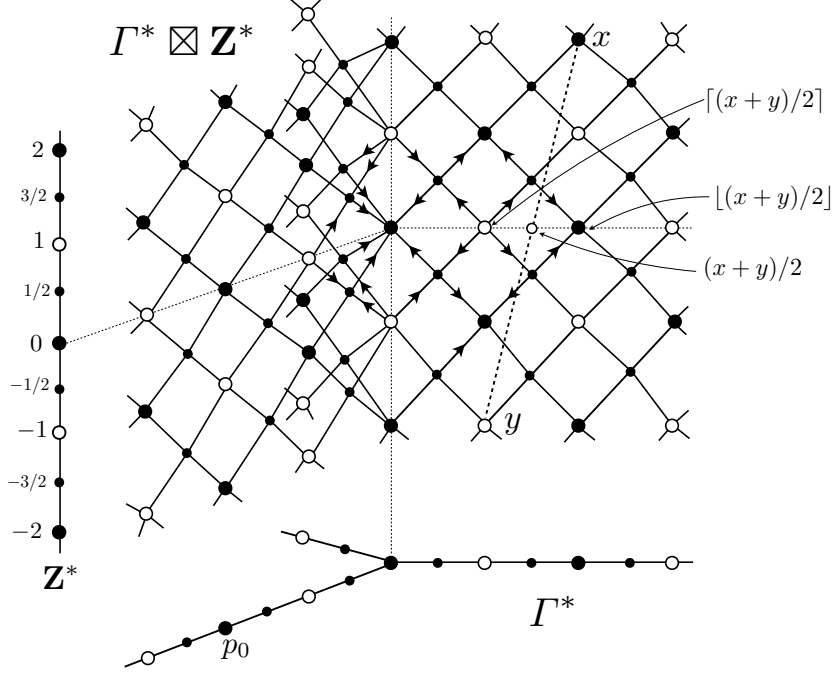
Figure 2: Graph $\Gamma^* \boxtimes \mathbf{Z}^*$

must be $(v_s, v_s, \ldots, v_s, v_0, v_t, \ldots, v_t)$. If $l \neq l'$, say $l < l'$, then $(p_{j_0}, p_{j_1}, \ldots, p_{j_m})$ must be $(v_s, \ldots, v_s, \bar{v}_s, v_0, \ldots, v_0, \bar{v}_t, v_t, \ldots, v_t)$ or $(v_s, \ldots, v_s, \bar{v}_s, \bar{v}_t, v_t, \ldots, v_t)$. Thus we obtain (o1). Since $\tilde{r}(i) = 0$ implies $r(i) = 0$, we obtain (o3). Hence (p, r) is an optimal potential, and f is a maximum multiflow.

3.2 Discrete convexity and steepest descent algorithm (SDA)

Here we briefly introduce a class of discrete convex functions (L-convex functions) on a certain graph structure and the steepest descent algorithm to minimize them. We then explain that our problem falls into the minimization of an L-convex function. The full discussion will be given in [18].

First we equip the space of all potentials with a graph structure. Let $\mathbf{Z}^*(:= \mathbf{Z}/2)$ denote the set of half-integers. Let $\Gamma^* \boxtimes \mathbf{Z}^*$ denote the set of pairs $(p, r) \in \Gamma^* \times \mathbf{Z}^*$ such that $p \in \Gamma$ if and only if $r \in \mathbf{Z}$. Two points (p, r) and (p', r') are adjacent if and only if p and p' are adjacent in Γ^* and $|r - r'| = 1/2$. Fix an arbitrary vertex p_0 of Γ. Let B (resp. W) denote the subset of $\Gamma^* \boxtimes \mathbf{Z}^*$ consisting of pairs (p, r) with $p \in \Gamma$, $r \in \mathbf{Z}$, and $d(p, p_0) + r$ even (resp. odd). Orient each edge of $\Gamma^* \boxtimes \mathbf{Z}^*$ by $(p, r) \leftarrow (p', r')$ if $(p, r) \in B$ or $(p', r') \in W$. Namely B is the set of sinks and W is the set of sources. This orientation is acyclic, and induces a partial order $\preceq$ on $\Gamma^* \boxtimes \mathbf{Z}^*$. See Figure 2, where nodes in B and W are colored black and white, respectively.

Next we define midpoint operations on $\Gamma^* \boxtimes \mathbf{Z}^*$. Let Γ^{**} denote the edge-subdivision of Γ^* with edge-length $1/4$, let $\mathbf{Z}^{**}(:= \mathbf{Z}/4)$ denote the set of quarter-integers, and let $\Gamma^{**} \boxtimes \mathbf{Z}^{**}$ denote the set of pairs $(p, r) \in \Gamma^{**} \times \mathbf{Z}^{**}$ such that $p \in \Gamma^*$ if and only if $r \in \mathbf{Z}^*$. For two points $x = (p, r), x' = (p', r')$ in $\Gamma^* \boxtimes \mathbf{Z}^*$, there exists a unique *midpoint* $y = (q, t) \in \Gamma^{**} \boxtimes \mathbf{Z}^{**}$ such that $d(p, q) + d(q, p') = d(p, p')$, $d(p, q) = d(q, p')$, and $t = (r + r')/2$. This y is denoted by $(x + x')/2$; accordingly q is denoted by $(p + p')/2$. For $z = (q, t) \in \Gamma^{**} \boxtimes \mathbf{Z}^{**}$, there uniquely exists a pair (x, y) of vertices in $\Gamma^* \boxtimes \mathbf{Z}^*$ with the property that $z = (x + y)/2$ and $x \preceq y$. We denote x and y by $\lfloor z \rfloor$ and $\lceil z \rceil$, respectively.

We are ready to define L-convex functions. For a natural number n, consider the product $(\Gamma^* \boxtimes \mathbf{Z}^*)^n$; a point x in $(\Gamma^* \boxtimes \mathbf{Z}^*)^n$ is represented by a pair (p, r) of $p \in (\Gamma^*)^n$ and $r \in (\mathbf{Z}^*)^n$. A function $g : (\Gamma^* \boxtimes \mathbf{Z}^*)^n \to \mathbf{R} \cup \{\infty\}$ is *L-convex* if it satisfies the following analogue of the *discrete midpoint convexity* [22, Section 7.2]:

$$g(x) + g(y) \geq g(\lfloor (x+y)/2 \rfloor) + g(\lceil (x+y)/2 \rceil) \quad (x, y \in (\Gamma^* \boxtimes \mathbf{Z}^*)^n), \tag{3.4}$$

where $(\lfloor (x+y)/2 \rfloor)_i := \lfloor (x_i + y_i)/2 \rfloor$ and $(\lceil (x+y)/2 \rceil)_i := \lceil (x_i + y_i)/2 \rceil$ for $i = 1, 2, \ldots, n$.

For $x \in (\Gamma^* \boxtimes \mathbf{Z}^*)^n$, let $\mathcal{F}_x$ (resp. $\mathcal{I}_x$) denote the set of points y with $x_i \preceq y_i$ (resp. $x_i \succeq y_i$) for $i = 1, 2, \ldots, n$. The set $\mathcal{F}_x \cup \mathcal{I}_x$ is called the *neighborhood* of x. The *steepest descent algorithm* is given as follows:

Algorithm 2: Steepest descent algorithm (SDA)

Input: An L-convex function $g : (\Gamma^* \boxtimes \mathbf{Z}^*)^n \to \mathbf{R} \cup \{\infty\}$, and a point x^0 in dom g.

Output: A minimizer of g.

Step 0: Let $i \leftarrow 0$.

Step 1: Find a minimizer y of g over the neighborhood $\mathcal{F}_{x^i} \cup \mathcal{I}_{x^i}$ of x^i.

Step 2: If $g(x^i) = g(y)$, then output x^i and stop; x^i is a minimizer.

Step 3: Otherwise, let $x^{i+1} \leftarrow y$, $i \leftarrow i + 1$, and go to step 1.

The fact that the output is a minimizer easily follows from (3.4); see [17, Theorem 2.5].

We discuss the number of iterations of this algorithm. For $x, y \in (\Gamma^* \boxtimes \mathbf{Z}^*)^n$, an l_∞-*path* between x and y is a sequence $P = (x = x^0, x^1, \ldots, x^m = y)$ such that for each k and i, the i-th components x_i^k and x_i^{k+1} belong to a common 4-cycle of $\Gamma^* \boxtimes \mathbf{Z}^*$, where m is called the *length* of P. The l_∞-*distance* between x and y, denoted by $D_\infty(x, y)$, is defined as the minimum length of an l_∞-path between x and y. For distinct $(p, r), (q, s) \in \Gamma^* \boxtimes \mathbf{Z}^*$ in a 4-cycle, it holds $d(p, q) + |r - s| = 1$. From this we observe that

$$D_\infty(x, y) = \max_{1 \leq i \leq n} d(p_i, q_i) + |r_i - s_i| \quad (x = (p, r), y = (q, s) \in (\Gamma^* \boxtimes \mathbf{Z}^*)^n). \tag{3.5}$$

Notice that a sequence $(x = x^0, x^1, x^2, \ldots, x^m)$ generated by SDA is an l_∞-path. Let opt(g) denote the set of all minimizers of g. The length m, which is the number of the iterations, is at least $D_\infty(x_0, \text{opt}(g)) = \min_{y \in \text{opt}(g)} D_\infty(x_0, y)$. This lower bound is almost tight.

Theorem 3.5 ([18]). *Let $(x = x^0, x^1, \ldots, x^m)$ be a sequence of points generated by SDA applied to an L-convex function g and an initial point x. Then $m \leq D_\infty(x, \text{opt}(g)) + 2$. If $g(x) = \min_{y \in \mathcal{F}_x} g(y)$ or $g(x) = \min_{y \in \mathcal{I}_x} g(y)$, then $m = D_\infty(x, \text{opt}(g))$.*

A similar bound for original L-convex functions in DCA was established in [23]. For a similar but different class of L-convex functions (called *alternating L-convex functions*), the same bound was proved by [17]. By using this result, we give a shorter proof of Theorem 3.5 in Appendix.

We now return to our problem. Let a pair of $N = (V, E, S, c, a)$ and $\mathcal{E} = (\Gamma, \{q_s\}_{s \in S})$ be an instance. Let $V = \{1, 2, \ldots, n\}$. Then the set of potentials $(p, r) \in (\Gamma^*)^V \times (\mathbf{Z}_+/2)^V$ is naturally regarded as a subset of $(\Gamma^* \boxtimes \mathbf{Z}^*)^n$. Define a function $g_{N,\mathcal{E}} : (\Gamma^* \boxtimes \mathbf{Z}^*)^n \to \mathbf{R} \cup \{\infty\}$ by

$$g_{N,\mathcal{E}}(x) := \sum_{i \in V \setminus S} 2c(i)r(i) \tag{3.6}$$

if $x = (p, r)$ is a potential and $g_{N,\mathcal{E}}(x) := \infty$ otherwise.

Proposition 3.6 ([18]). *Suppose that a is even-valued. Then $g_{N,\mathcal{E}}$ is L-convex.*

In Appendix, we give a shorter proof of this proposition for our case where a is positive.

4 Algorithm

In this section, we prove the main result (Theorem 1.1) by presenting an algorithm (*dual descent algorithm*) to solve a nondegenerate instance of our multiflow problem. We first show that the optimality check of a potential (p, r), or finding a (p, r)-admissible support, is reduced to the submodular flow feasibility problem on a certain skew-symmetric network, called the *double covering network*. This extends the earlier result on the minimum-cost edge-capacitated multiflow problem by Karzanov [19, 20], in which the optimality is checked by the classical circulation problem. Partial adaptations of this idea to the node-capacitated setting have been given in [3, 4]; but the full adaptation using submodular flow is new. Checking the feasibility of a submodular flow is reduced to the maximum submodular flow problem. We prove that the minimal minimum cut naturally gives a steepest direction at each potential. Then we obtain a simple descent algorithm mentioned in Introduction.

4.1 Double covering network with submodular constraints

Let a pair of $N = (V, E, S, c, a)$ and $\mathcal{E} = (\Gamma, \{q_s\}_{s \in S})$ be a nondegenerate instance. We can assume that there is no edge such that both of its ends are terminals. Let (p, r) a potential for $(N, \mathcal{E})$. A nonterminal node i is said to be *flat* if $p(i) \notin \Gamma_3$, and *singular* if $p(i) \in \Gamma_3$. A nonterminal node i is said to be *zero* if $r(i) = 0$ and *positive* if $r(i) > 0$. We are going to construct a skew-symmetric network $\mathcal{D}_{p,r}$ associated with (p, r). We first define a signed node set U_i and edge set A_i indexed by each node i in N, together with lower edge-capacity $\underline{c}$ and upper edge-capacity $\overline{c}$. For each terminal s, let $U_s := \{s^+, s^-\}$, and let $A_s := \{s^+ s^-\}$ with $\underline{c}(s^+ s^-) := 0$ and $\overline{c}(s^+ s^-) := \infty$. For each flat node i, let $U_i := \{i_1^+, i_2^+, i_1^-, i_2^-\}$, and let $A_i := \{i_1^+ i_2^-, i_2^+ i_1^-\}$ with $\overline{c}(i_1^+ i_2^-) = \overline{c}(i_2^+ i_1^-) := c(i)$. If i is positive, then $\underline{c}(i_1^+ i_2^-) = \underline{c}(i_2^+ i_1^-) := c(i)$. Otherwise $\underline{c}(i_1^+ i_2^-) = \underline{c}(i_2^+ i_1^-) := 0$. For each positive singular node i, let $U_i := \{i_0^+, i_1^+, i_2^+, i_3^+, i_0^-, i_1^-, i_2^-, i_3^-\}$ and let A_i consist of $i_0^+ i_0^-$, $i_k^+ i_0^+$, $i_0^- i_k^-$ $(k = 1, 2, 3)$ with $\underline{c}(i_0^+ i_0^-) = \overline{c}(i_0^+ i_0^-) := 2c(i)$, $\underline{c}(i_k^+ i_0^+) = \underline{c}(i_0^- i_k^-) := 0$, and $\overline{c}(i_k^+ i_0^+) = \overline{c}(i_0^- i_k^-) := c(i)$ $(k = 1, 2, 3)$. For each zero singular node i, let $U_i := \{i_1^+, i_2^+, i_3^+, i_1^-, i_2^-, i_3^-\}$ and let $A_i := \emptyset$. Define a submodular function Δ_i^* on U_i by $\Delta_i^*(X) := \Delta_{c(i)}^*(X')$, where X' is obtained from X by replacing each i_k^+ (resp. i_k^-) with k^+ (resp. k^-). For an edge $e = ij \in E_{p,r}$ with $ij \in \delta_{p,k}(i)$ and $ij \in \delta_{p,k'}(j)$, add edges $e^+ = i_k^- j_{k'}^+$ and $e^- = j_{k'}^- i_k^+$ with $\underline{c}(e^+) = \underline{c}(e^-) := 0$ and $\overline{c}(e^+) = \overline{c}(e^-) := \infty$. In the case where j is a terminal s, let $e^+ = i_k^- s^+$ and $e^- = s^- i_k^+$. Now the *double covering network* $\mathcal{D}_{p,r} = (U, A, \underline{c}, \overline{c})$ is defined by the disjoint union of all these edges and nodes. See Figure 3, where i, j, i', and j' are positive singular, zero singular, zero flat, and positive flat nodes, respectively.

A node v in $\mathcal{D}_{p,r}$ is said to be *unusual* if v belongs to U_i for a zero singular node i. Other nodes v are said to be *usual*.

A *circulation* φ in $\mathcal{D}_{p,r}$ is a function on edge set A such that

$$\underline{c}(e) \leq \varphi(e) \leq \overline{c}(e) \quad (e \in A),$$
$$(\nabla \varphi)(v) = 0 \quad (\text{usual node } v \text{ in } \mathcal{D}_{p,r}),$$
$$(\nabla \varphi)|_{U_i} \in \mathcal{B}(\Delta_i^*) \quad (\text{zero singular node } i \text{ in } N).$$

For a circulation φ, define $\zeta_\varphi : E_{p,r} \to \mathbf{R}_+$ by

$$\zeta_\varphi(e) := \frac{\varphi(e^+) + \varphi(e^-)}{2} \quad (e \in E_{p,r}). \tag{4.1}$$

Lemma 4.1. *Suppose that an integral circulation φ in $\mathcal{D}_{p,r}$ exists. Then ζ_φ is a (p, r)-admissible support.*
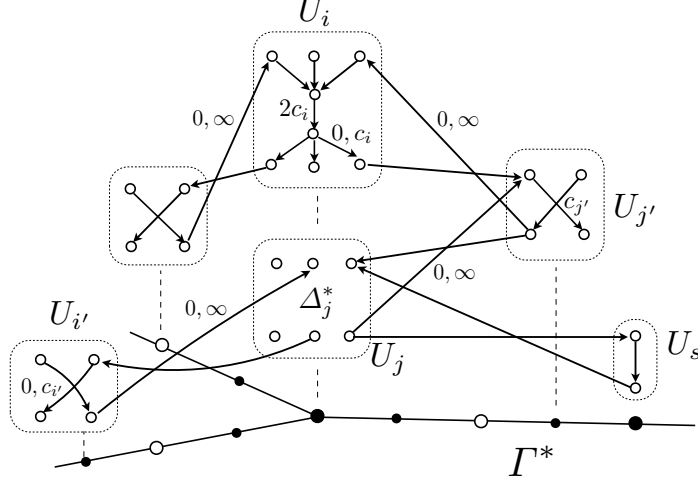
Figure 3: Double covering network

Proof. The half-integrality of ζ_φ (in (a4)) is clear. Take a nonterminal node i. Let $\zeta_k := \zeta_\varphi(\delta_{p,k}(i)) = \sum_{e \in \delta_{p,k}(i)}(\varphi(e^+) + \varphi(e^-))/2$. Suppose that i is flat. Then $\zeta_1 = \sum_{e \in \delta_{p,1}(i)}(\varphi(e^+) + \varphi(e^-))/2 = (\varphi(i_1^+ i_2^-) + \varphi(i_2^+ i_1^-))/2 = \sum_{e \in \delta_{p,2}(i)}(\varphi(e^+) + \varphi(e^-))/2 = \zeta_2$. Since $\varphi(i_1^+ i_2^-) \leq c(i)$ and $\varphi(i_2^+ i_1^-) \leq c(i)$. We obtain (a1). In addition, if i is positive, then $\varphi(i_1^+ i_2^-) = \varphi(i_2^+ i_1^-) = c(i)$ must hold, and we obtain (a3). Since ζ_1 and ζ_2 are half-integers with $\zeta_1 = \zeta_2$, we have (a4).

Suppose that i is positive singular. Since $\zeta_k = \sum_{e \in \delta_{p,k}(i)}(\varphi(e^+) + \varphi(e^-))/2 = (\varphi(i_k^+ i_0^+) + \varphi(i_0^- i_k^-))/2 \leq c(i)$, we obtain $\zeta_k \leq c(i)$. Moreover it holds $\zeta_\varphi(\delta\{i\}) = \zeta_1 + \zeta_2 + \zeta_3 = \varphi(i_0^+ i_0^-) = 2c(i)$, implying (a3) and (a4). If $\zeta_1 > \zeta_2 + \zeta_3$, then $\zeta_1 + \zeta_2 + \zeta_3 < 2\zeta_1 \leq 2c(i)$; this contradicts $\zeta_1 + \zeta_2 + \zeta_3 = 2c(i)$. Therefore ζ_1, ζ_2, and ζ_3 satisfy (a2).

Suppose that i is zero singular. Notice that $(\nabla\varphi)(i_k^+) = \sum_{e \in \delta_{p,k}(i)} \varphi(e^+)$, and $(\nabla\varphi)(i_k^-) = -\sum_{e \in \delta_{p,k}(i)} \varphi(e^-)$. Thus $(\zeta_1, \zeta_2, \zeta_3) = \phi((\nabla\varphi)|_{U_i})$; see (2.5) for ϕ. Since $(\nabla\varphi)|_{U_i} \in \mathcal{B}(\Delta_i^*)$, because of Lemmas 2.3 and 2.5, the vector $(\zeta_1, \zeta_2, \zeta_3)$ satisfies (a2). Since $0 = (\nabla\varphi)(U_i) = \sum_{k=1,2,3}(\nabla\varphi)(i_k^+) + \sum_{k=1,2,3}(\nabla\varphi)(i_k^-)$, we have $\sum_{k=1,2,3}(\nabla\varphi)(i_k^+) = -\sum_{k=1,2,3}(\nabla\varphi)(i_k^-)$ and obtain (a4) by

$$\zeta_\varphi(\delta\{i\}) = \zeta_1 + \zeta_2 + \zeta_3 = \frac{1}{2}\sum_{k=1,2,3}(\nabla\varphi)(i_k^+) - (\nabla\varphi)(i_k^-) = \sum_{k=1,2,3}(\nabla\varphi)(i_k^+) \in \mathbf{Z}.$$

□

4.2 Dual descent algorithm: implementing SDA by submodular flow

To check the existence of a circulation in $\mathcal{D}_{p,r}$, we construct an instance of the maximum submodular flow problem. Add a super source a^+ and super sink a^-. For each edge $e = v^+ u^-$ in $\mathcal{D}_{p,r}$ having nonzero lower capacity $\underline{c}(e) > 0$, replace $v^+ u^-$ by two edges $v^+ a^-$ and $a^+ u^-$ with (upper) capacity $\underline{c}(e)$ (and lower capacity 0). Those edges are $i_0^+ i_0^-$ for positive singular nodes i and $i_1^+ i_2^-, i_2^+ i_1^-$ for positive flat nodes i. The resulting (skew-symmetric) network is denoted by $\tilde{\mathcal{D}}_{p,r}$, where modified edge sets are denoted by $\tilde{A}_i$ ($i \in V$) and the (upper) edge-capacity is denoted by $\tilde{c}$. Consider the maximum (a^+, a^-)-submodular flow problem on $\tilde{\mathcal{D}}_{p,r}$, where submodular function ρ on U is given as

$$\rho(X) := \sum_{i:\text{zero singular}} \Delta_i^*(X \cap U_i) \quad (X \subseteq U). \tag{4.2}$$

17

Lemma 4.2. *If $\{a^+\}$ is a minimum (a^+, a^-)-cut in $\tilde{\mathcal{D}}_{p,r}$, then a circulation φ in $\mathcal{D}_{p,r}$ exists, and is obtained from any maximum flow φ' by the following procedure:*

(A) *For each edge $e = v^+ u^-$ in $\mathcal{D}_{p,r}$ having nonzero lower capacity, let $\varphi(e) := \varphi'(a^+ u^-)$, and for other edge e in $\mathcal{D}_{p,r}$, let $\varphi(e) := \varphi'(e)$.*

Indeed, in this case, any max-flow saturates all edges leaving a^+ and all edges entering a^-, and consequently the resulting φ satisfies $\varphi(e) = \underline{c}(e) = \bar{c}(e)$ for all edges e having nonzero lower-capacity, and is a circulation of $\mathcal{D}_{p,r}$.

Next we show that the minimal minimum cut gives rise to a steepest descent direction of $g_{N,\mathcal{E}}$ at (p, r). An (a^+, a^-)-cut X is said to be *normal* if it satisfies the following conditions:

(c0) X does not meet $\{s^+, s^-\}$ for any terminal $s \in S$.

(c1) X is a transversal.

(c2) For each positive node i, $X \cap U_i^+$ is equal to $\emptyset$, U_i^+, or $\{i_k^+\}$, and $X \cap U_i^-$ is equal to $\emptyset$, U_i^-, or $U_i^- \setminus \{i_k^+\}$.

(c3) For each zero node i, $X \cap U_i$ is equal to $\emptyset$, U_i^+, $\{i_k^+\}$, or $\{i_k^+\} \cup U_i^- \setminus \{i_k^-\}$.

Then the following holds; the proof is given in the end of this subsection.

Lemma 4.3. *A normal minimum (a^+, a^-)-cut exists, and is obtained from the unique minimal minimum cut X by applying the following procedure:*

(B) *For each singular node i, if $|X \cap U_i^+| = 2$, then replace X by $X \cup U_i^+$.*

Let $U_{\mathcal{I}}$ be the subset of U consisting of i_k^+, i_k^- for $i \in V \setminus S$ and $k \in \{1, 2, 3\}$ such that $(p(i), r(i)) \in W$, or $p(i) \in \Gamma^* \setminus \Gamma$ and $(p(i)_{\to^* k}, r(i) - 1/2) \in W$. Let $U_{\mathcal{F}}$ be the node subset defined by replacing W with B in the definition of $U_{\mathcal{I}}$. A normal cut X is said to be $\mathcal{F}$-*normal* if $X \cap U_{\mathcal{I}} = \emptyset$, and is said to be *of* $\mathcal{I}$-*normal* if $X \cap U_{\mathcal{F}} = \emptyset$. For an $\mathcal{F}$-normal or $\mathcal{I}$-normal cut X, define $(p, r)^X$ by

$$
(p, r)^X(i) := \begin{cases}
(p(i)_{\to^* k}, r(i) + 1/2) & \text{if } X \cap U_i = \{i_k^+\}, \\
(p(i)_{\to^* k}, r(i) - 1/2) & \text{if } X \cap U_i = U_i^- \setminus \{i_k^-\}, \\
(p(i)_{\to k}, r(i)) & \text{if } X \cap U_i = \{i_k^+\} \cup U_i^- \setminus \{i_k^-\}, \\
(p(i), r(i) + 1) & \text{if } X \cap U_i = U_i^+, \\
(p(i), r(i) - 1) & \text{if } X \cap U_i = U_i^-, \\
(p(i), r(i)) & \text{if } X \cap U_i = \emptyset
\end{cases}
\tag{4.3}
$$

for each nonterminal node $i \in V \setminus S$ and $(p, r)^X(s) := (q_s, 0)$ for each terminal $s \in S$. See Figure 4 for the correspondence between $(p, r)^X(i)$ and $U_i \cap X$, where nodes in $U_i \cap X$ are colored black.

Proposition 4.4. *Let X be a normal minimum (a^+, a^-)-cut in $\tilde{\mathcal{D}}_{p,r}$, and let $X_{\mathcal{F}} := X \cap U_{\mathcal{F}}$ and $X_{\mathcal{I}} := X \cap U_{\mathcal{I}}$. Then $(p, r)^{X_{\mathcal{F}}}$ is a minimizer of $g_{N,\mathcal{E}}$ over $\mathcal{F}_{p,r}$, and $(p, r)^{X_{\mathcal{I}}}$ is a minimizer of $g_{N,\mathcal{E}}$ over $\mathcal{I}_{p,r}$.*

The proof is given in the end. We are now ready to describe our algorithm.

Algorithm 3: Dual descent algorithm (for a nondegenerate instance)

Input: A nondegenerate instance $N = (V, E, S, c, a)$, $\mathcal{E} = (\Gamma, \{q_s\}_{s \in S})$.

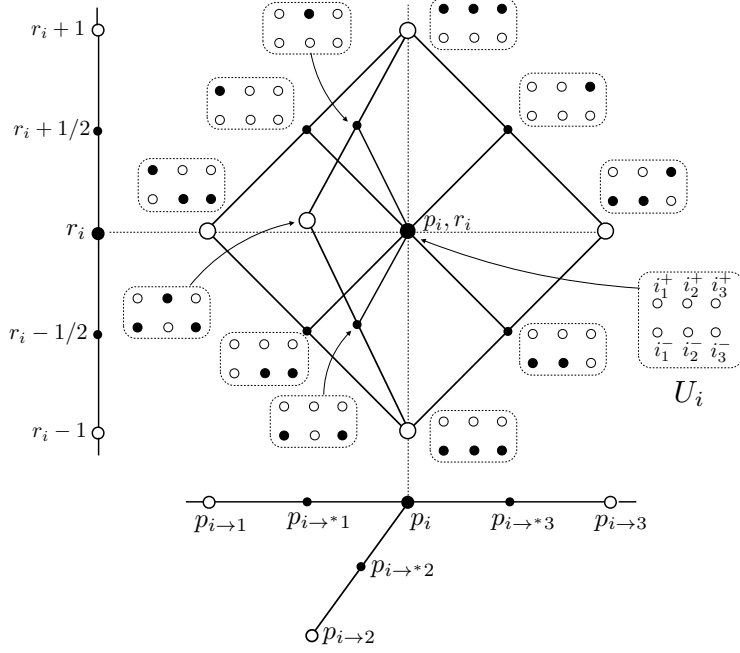**Output:** A half-integral optimal multiflow f.

Figure 4: Neighborhood and normal cut

Step 0: Choose any vertex v in Γ_0, and let (p, r) be a potential defined by $(p(i), r(i)) :=$ $(v, d(\Gamma_0))$ for $i \in V \setminus S$ and $(p(s), r(s)) := (q_s, 0)$ for $s \in S$.

Step 1: Construct network $\tilde{\mathcal{D}}_{p,r}$. Obtain an integral maximum (a^+, a^-)-flow φ' and the minimal minimum (a^+, a^-)-cut X in $\tilde{\mathcal{D}}_{p,r}$ by a submodular flow algorithm. Make X normal by Procedure (B).

Step 2: If $X = \{a^+\}$, then (p, r) is optimal, obtain a feasible flow φ in $\mathcal{D}_{p,r}$ from φ' by Procedure (A). obtain a (p, r)-admissible support ζ_φ by (4.1), and obtain a half-integral optimal multiflow f from ζ_φ by Algorithm 1; stop.

Step 3: Choose $Y \in \{X_{\mathcal{F}}, X_{\mathcal{I}}\}$ with $g_{N,\mathcal{E}}((p, r)^Y) = \min\{g_{N,\mathcal{E}}((p, r)^{X_{\mathcal{F}}}), g_{N,\mathcal{E}}((p, r)^{X_{\mathcal{I}}})\}$. Let $(p, r) \leftarrow (p, r)^Y$, and go to step 1.

Theorem 4.5. *The dual descent algorithm runs in $O(d(\Gamma_0)\mathrm{MSF}(n, m, 1))$ time.*

Proof. By Proposition 4.4, $(p, r)^Y$ is a steepest direction of $g_{N,\mathcal{E}}$ at (p, r). Hence the dual descent algorithm is viewed as the steepest descent algorithm applied to $g_{N,\mathcal{E}}$ that is L-convex (Proposition 3.6). By Lemma 3.4 and Theorem 3.5, the number of iteration is bounded by $d(\Gamma_0)$. Our submodular function ρ is a disjoint sum of submodular functions Δ_i^* for zero singular nodes i (see (4.2)). So the exchange capacity $\kappa(\cdot; u, v)$ for a pair of u and v can take positive if u, v belong to U_i for some zero singular node i, and is equal to the exchange capacity for submodular function Δ_i^* on a 6-element set U_i. Hence this can be computed in constant time. The number of nodes of $\tilde{\mathcal{D}}_{p,r}$ is at most $6n + 2$, and the number of edges is at most $2m + 8n$. Thus step 2 is done in $O(\mathrm{MSF}(n, m, 1))$ time. $\square$

Proof of the main result (Theorem 1.1). As in Section 3, construct nondegenerate instance $(\tilde{N}, \tilde{\mathcal{E}})$. Then $d(\Gamma_0) \leq O(m \log k)$. Apply the dual descent algorithm for $(\tilde{N}, \tilde{\mathcal{E}})$. By Theorem 4.5, we obtain an optimal potential $(\tilde{p}, \tilde{r})$ and an optimal multiflow

f in $O((m \log k)\mathrm{MSF}(n, m, 1))$ time. As we have shown in Section 3, f is also a maximum multiflow, and an optimal potential (p, r) for the original instance is obtained from $(\tilde{p}, \tilde{r})$ (in $O(nm \log k)$ time). Then r is a half-integral optimal solution of LP-dual (1.2). Indeed, $\sum_{i \in V \setminus S} c(i) r(i)$ is equal to the maximum flow-value. The feasibility of r follows from $\sum_{i \in V(P) \setminus S} 2r(i) = \sum_{ij \in E(P)} (r(i) + r(j)) \geq \sum_{ij \in E(P)} d(p_i, p_j) \geq 2$ for every S-path P.

Proof of Lemma 4.3. Let X be the unique minimal minimum (a^+, a^-)-cut (of finite cut capacity). By Lemma 2.2, X is a transversal, implying (c1). Also X cannot meet $\{s^+, s^-\}$ for any terminal s. Otherwise, X contains s^- (and does not contain s^+). However the deletion of s^- from X does not increase the cut capacity, contradicting the minimality.

Consider a zero flat node i. Suppose that $i_k^- \in X$. If $i_{k'}^+ \notin X$ ($k' \neq k$), then the deletion of i_k^- from X decreases the capacity, contradicting the minimality. Thus we have (c3).

Consider a zero singular node i. Suppose that $X \cap U_i \neq \emptyset$. If $X \cap U_i^- = \{i_k^-\}$, then $X \cap U_i$ is of type 1, 3, or 5, the change $X \to X \setminus \{i_k^-\}$ preserves the type at U_i, and the cut capacity; contradicting the minimality. Similarly, if $X \cap U_i \subseteq U_i^-$, then the change $X \to X \setminus U_i^-$ preserves the cut capacity; a contradiction. Thus $X \cap U_i \nsubseteq U_i^-$. So suppose that $U_i^- \cap X = \emptyset$ or $U_i^- \setminus \{i_k^-\}$. If $U_i^- \cap X = U_i^- \setminus \{i_k^-\}$, then necessarily $U_i^+ = \{i_k^+\}$. Suppose that $U_i^- \cap X = \emptyset$. If $|X \cap U_i^+| \geq 2$ (type 1), then the change $X \to X \cup U_i^+$ (Procedure (B)) does not increase the cut capacity (and keeps X being a transversal). In particular, the resulting $X \cap U_i$ is one of the patterns in (c3).

Consider a positive singular node i. First we show that $|X \cap \{i_1^-, i_2^-, i_3^-\}| \neq 1$. Suppose to the contrary that $X \cap U_i = \{i_1^-\}$. The the change $X \to X \setminus \{i_1^-, i_0^-\}$ preserves the capacity; a contradiction to the minimality. Also $X \cap U_i^- = \{i_0^-\}$ is impossible, and $|X \cap \{i_1^-, i_2^-, i_3^-\}| \geq 2$ implies $i_0^- \in X$. Thus the pattern of $X \cap U_i^-$ is one given in (c2). Next consider $X \cap U_i^+$. If X contains two nodes in $\{i_1^+, i_2^+, i_3^+\}$, and necessarily $X \cap U_i^- = \emptyset$; the change $X \to X \cup U_i^+$ (Procedure (B)) keeps X being a transversal, and does not increase the cut capacity. If X contains i_0^+, then X contains at least two nodes in U_i^+, reduced to the case above. Thus, after Procedure (B), the resulting cut is a normal minimum cut, as required.

Proof of Proposition 4.4. In $\Gamma^* \boxtimes \mathbf{Z}^*$, vertices in B are colored black, and vertices in W are colored white. Other vertices have no color. For $(p, r) \in \Gamma^* \boxtimes \mathbf{Z}^*$ and $p' \in \Gamma^*$ with $p \neq p'$, let $(p, r)_{\searrow p'} := (u, r - 1/2)$ for the unique neighbor u of p with $d(p, p') = d(u, p') + 1/2$. In addition, if $(p, r) \notin B \cup W$, then let $(p, r)_{\nearrow p'} := (u, r - 1/2)$ for the (unique) neighbor u of p with $d(p, p') = d(u, p') - 1/2$.

Lemma 4.6. *For $(p, r), (p', r') \in \Gamma^* \boxtimes \mathbf{Z}^*$ with $d(p, p') \geq 1$, $d(p, p') - r - r'$ is even if and only if one of following pairs has the same color:*

$$((p, r), (p', r')), \quad ((p, r)_{\searrow p'}, (p', r')), \quad ((p, r), (p', r')_{\searrow p}), \quad ((p, r)_{\searrow p'}, (p', r')_{\searrow p}).$$

Proof. For $(p, r), (p', r') \in B \cup W$, observe that $d(p, p') - r - r' \equiv d(p, p_0) + r + d(p', p_0) + r'$ mod 2. Hence $d(p, p') - r - r'$ is even if and only if (p, r) and (p', r') have the same color. Also observe that $d(p, p') - r - r'$ does not change when (p, r) is replaced by $(p, r)_{\searrow p'}$. The claim follows from these facts. $\square$

Lemma 4.7. *There is no edge between $U_{\mathcal{F}}$ and $U_{\mathcal{I}}$. In particular, any normal cut X is decomposed into $\mathcal{F}$-normal cut $X_{\mathcal{F}} := X \cap U_{\mathcal{F}}$ and $\mathcal{I}$-normal cut $X_{\mathcal{I}} := X \cap U_{\mathcal{I}}$ such that*

$$\tilde{c}(\delta X) - \tilde{c}(\delta\{a^+\}) = \tilde{c}(\delta X_{\mathcal{F}}) - \tilde{c}(\delta\{a^+\}) + \tilde{c}(\delta X_{\mathcal{I}}) - \tilde{c}(\delta\{a^+\}).$$

Proof. Edge $i_k^+ i_{k'}^-$ (or $i_k^- i_{k'}^+$) in $\tilde{\mathcal{D}}_{p,r}$ belongs to U_i for node i of colored (p_i, r_i), and hence belongs to $U_{\mathcal{F}}$ or $U_{\mathcal{I}}$. So consider an edge $i_k^- j_{k'}^+$ for distinct i, j. Then $d(p_i, p_j) > 1$ (since

$a_{ij} \geq 2$) and $d(p_i, p_j) - r_i - r_j$ is even (since a_{ij} is even). By the previous lemma, if both (p_i, r_i) and (p_j, r_j) are colored, then they have the same color. Suppose that (p_i, r_i) has no color. If (p_j, r_j) has a color, then $(p_i, r_i)_{\searrow p_j} = (p_{i \to^* k}, r_i - 1/2)$ has the same color. If (p_j, r_j) has no color, then $(p_i, r_i)_{\searrow p_j} = (p_{i \to^* k}, r_i - 1/2)$ and $(p_j, r_j)_{\searrow p_i} = (p_{j \to^* k'}, r_j - 1/2)$ have the same color. Consequently $i_k^-, j_{k'}^+ \in U_{\mathcal{F}}$ or $i_k^-, j_{k'}^+ \in U_{\mathcal{I}}$ for all cases. □

Let $\mathcal{F}_{p,r}^+$ (resp. $\mathcal{I}_{p,r}^+$) denote the subset of $\mathcal{F}_{p,r}$ (resp. $\mathcal{I}_{p,r}$) consisting (p, r) with $r_i \geq 0$ for $i = 1, 2, \ldots, n$. Proposition 4.4 follows from the above Lemma 4.7 and the following.

Lemma 4.8. (1) *The map $X \to (p, r)^X$ is a bijection between $\mathcal{F}_{p,r}^+$ (resp. $\mathcal{I}_{p,r}^+$) and the set of all $\mathcal{F}$-normal cuts (resp. $\mathcal{I}$-normal cuts).*

(2) *For an $\mathcal{F}$-normal or $\mathcal{I}$-normal cut X, it holds*

$$g_{N,\mathcal{E}}((p, r)^X) - g_{N,\mathcal{E}}(p, r) = \tilde{c}(\delta X) + \rho(X \setminus \{a^+\}) - \tilde{c}(\delta\{a^+\}). \tag{4.4}$$

Proof. (1). We claim that $\mathcal{X}_i := \{X \cap U_i \mid X : \mathcal{F}\text{-normal cut}\}$ and $\mathcal{F}_{p_i, r_i}^+$ are in one-to-one correspondence by (4.3). Suppose that $(p_i, r_i) \in W$, which implies $U_i \subseteq U_{\mathcal{I}}$. Then $\mathcal{F}_{p_i, r_i}^+ = \mathcal{F}_{p_i, r_i} = \{(p_i, r_i)\}$ and $\mathcal{X}_i = \{\emptyset\}$. Thus the claim is true. Suppose that $(p_i, r_i) \in B$, which implies $U_i \subseteq U_{\mathcal{F}}$. The claim can be seen from Figure 4. Suppose that $(p_i, r_i) \notin B \cup W$; necessarily i is positive. We can assume that $(p_{i \to^* 1}, r_i + 1/2) \in B$ (and $(p_{i \to^* 1}, r_i - 1/2) \in W$). Then $\mathcal{F}_{p_i, r_i}^+ = \{(p_i, r_i), (p_{i \to^* 1}, r_i - 1/2), (p_{i \to^* 2}, r_i + 1/2)\}$. Since $i_1^+, i_1^- \in U_{\mathcal{I}}$ and $i_2^+, i_2^- \in U_{\mathcal{F}}$, it holds that $\mathcal{X}_i = \{\emptyset, \{i_2^-\}, \{i_2^+\}\}$. Thus we have the claim, implying the statement (1).

(2). Let $(p', r') := (p, r)^X$. We first show that $g(p', r') < \infty$ if and only if $\tilde{c}(\delta X)$ is finite. Suppose that $g(p', r') < \infty$. Pick $i_k^- \in X$ and edge $i_k^- j_{k'}^+$. We show $j_{k'}^+ \in X$. Recall that $d(p_i, p_j) - r_i - r_j - a_{ij} = 0$ and $a_{ij} \geq 2$. Hence $d(p_i, p_j) \geq 2$. Since $i_k^- \in X$, the change $(p_i, r_i) \to (p_i', r_i')$ increases $d(p_i, p_j) - r_i - r_j$ by one. Necessarily the change $(p_j, r_j) \to (p_j', r_j')$ must decrease $d(p_i', p_j) - r_i' - r_j$ by one. This means that $(p_j', r_j') = (p_j, r_j + 1)$, $(p_{j \to_{k'}}, r_j)$ or $(p_{j \to^* k'}, r_j + 1/2)$ must hold. For all cases, X contains $j_{k'}^+$.

Suppose that $\tilde{c}(\delta X)$ is finite. We show $g(p', r') < \infty$. Here $r' \geq 0$ is clear. We need to show that $d(p_i', p_j') - r_i' - r_j' - a_{ij} \leq 0$ for each edge $ij \in E$. Suppose that $d(p_i, p_j) - r_i - r_j - a_{ij} = 0$. Namely $ij \in E_{p,r}$ and $d(p_i, p_j) \geq 2$. In this case, the argument is the same as the above. Indeed, suppose that the change $(p_i, r_i) \to (p_i', r_i')$ increases $d(p_i, p_j) - r_i - r_j$ by one. Then X contains i_k^- with $p_j \in \Gamma_{p_i, k}^*$, and hence contains $j_{k'}^+$ with $p_i \in \Gamma_{p_j, k'}^*$. The change $(p_j, r_j) \to (p_j', r_j')$ decreases $d(p_i', p_j) - r_i' - r_j$ by one, implying $d(p_i', p_j') - r_i' - r_j' - a_{ij} = 0$.

Thus it suffices to show that $d(p_i', p_j') - r_i' - r_j' - a_{ij} = 1$ cannot occur. Otherwise, $d(p_i, p_j) - r_i - r_j - a_{ij} = -1$, $d(p_i, p_j) - r_i - r_j$ is odd, and $(p_i', r_i') \neq (p_i, r_i)$, $(p_j', r_j') \neq (p_j, r_j)$. If both (p_i, r_i) and (p_j, r_i) have colors, then these colors are different (by Lemma 4.6), and $(p_i', r_i') = (p_i, r_i)$ or $(p_j', r_j') = (p_j, r_j)$ must hold; a contradiction. Suppose that (p_i, r_i) has no color. Then $(p_i', r_i') = (p_i, r_i)_{\nearrow p_j}$ must hold. If (p_j', r_j') has a color, then this color must equal the color of (p_i', r_i') (since $(p_i, r_i)_{\nearrow p_j}$ and $(p_i, r_i)_{\searrow p_j}$ have different colors), and hence $(p_j', r_j') = (p_j, r_j)$; a contradiction. If (p_j', r_j') has no color, then $(p_j', r_j') = (p_j, r_j)_{\nearrow p_i}$ must hold; but this is impossible since colors of $(p_i, r_i)_{\nearrow p_j}$ and $(p_j, r_j)_{\nearrow p_i}$ are different.

Finally we show the equation (4.4). The left hand side is equal to $\sum_{i \in V \setminus S} 2c_i(r_i' - r_i)$ and the right hand side is equal to

$$\sum_{i:\text{positive}} \{\tilde{c}(\tilde{A}_i \cap \delta X) - 2c(i)\} + \sum_{i:\text{zero flat}} \tilde{c}(\tilde{A}_i \cap \delta X) + \sum_{i:\text{zero singular}} \Delta_i^*(X \cap U_i)$$

One can verify from the network construction (see Figure 3), definitions of $(p, r)^X$ (see (4.3)) and Δ_c^* (see (2.7)) that $2c_i(r_i' - r_i)$ is equal to $\tilde{c}(\tilde{A}_i \cap \delta X) - 2c(i)$ if i is positive, $\tilde{c}(\tilde{A}_i \cap \delta X)$ if i is zero flat, and $\Delta_i^*(X \cap U_i)$ if i is zero singular. Thus we obtain the equation (4.4). □

Acknowledgments

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A   Appendix

For $x \in \Gamma^* \boxtimes \mathbf{Z}^*$, the first and second components of x are denoted by x_p and x_r, respectively, i.e., $x = (x_p, x_r)$.

Proof of Theorem 3.5.   First consider the case where Γ is a path (with no ends). Then $\Gamma^* \boxtimes \mathbf{Z}^*$ is isomorphic to the product of two zigzagly-oriented paths. In this case, $(\Gamma^* \boxtimes \mathbf{Z}^*)^n$ is identified with the product $(\mathbf{Z}^*)^{2n}$ of $2n$ paths, and L-convex functions on $(\Gamma^* \boxtimes \mathbf{Z}^*)^n$ coincide with *alternating L-convex functions* in the sense of [17]. Also D_∞ is equal to d in [17]. Then Theorem 3.5 was shown in [17, Theorem 2.6]. In particular, the following holds:

Proposition A.1. *For $x \in \operatorname{dom} g$ and a minimizer x' of g over $\mathcal{I}_x \cup \mathcal{F}_x$ with $g(x') < g(x)$, if $g(x) = \min_{y \in \mathcal{I}_x} g(y)$ or $g(x) = \min_{y \in \mathcal{F}_x} g(y)$, then it holds*

$$D_\infty(x', \operatorname{opt}(g)) = D_\infty(x, \operatorname{opt}(g)) - 1. \tag{A.1}$$

We show that this proposition holds for a general tree Γ. Pick an arbitrary $z \in \operatorname{opt}(g)$ with $D_\infty(x, \operatorname{opt}(g)) = D_\infty(x, z)$. Since $(x_i')_p$ and $(x_i)_p$ are equal or adjacent in Γ or Γ^*, there is a path P_i in Γ^* containing $(x_i)_p, (x_i')_p$, and $(z_i)_p$. Then z, x, and x' are points in $\prod_{i=1}^n P_i \boxtimes \mathbf{Z}^* \simeq (\mathbf{Z}^*)^{2n}$, and the restriction g' of g to $\prod_{i=1}^n P_i \boxtimes \mathbf{Z}^*$ is (alternating) L-convex. Thus Proposition A.1 is applicable to g', x, x', and hence $D_\infty(x', \operatorname{opt}(g')) = D_\infty(x, \operatorname{opt}(g')) - 1 = D_\infty(x, \operatorname{opt}(g)) - 1$. By $D_\infty(x', \operatorname{opt}(g)) \leq D_\infty(x', \operatorname{opt}(g'))$, the equation (A.1) holds.

Theorem 3.5 is proved as follows. By the description of the steepest descent algorithm, it holds that $g(x^i) = \min_{y \in \mathcal{I}_{x^i}} g(y)$ or $g(x^i) = \min_{y \in \mathcal{F}_{x^i}} g(y)$ for all $i > 0$. Thus $m - 1 = D_\infty(x^1, \operatorname{opt}(g))$ holds. Since $D_\infty(x^1, \operatorname{opt}(g)) \leq D_\infty(x, \operatorname{opt}(g)) + 1$, we obtain $m \leq D_\infty(x, \operatorname{opt}(g)) + 2$. In addition, if $g(x) = \min_{y \in \mathcal{I}_x} g(y)$ or $g(x) = \min_{y \in \mathcal{F}_x} g(y)$, then $m = D_\infty(x, \operatorname{opt}(g))$ holds.

Proof of Proposition 3.6 for a positive cost. For an even positive integer $a(\geq 2)$, define $h : (\Gamma^{**} \boxtimes \mathbf{Z}^{**})^2 \to \mathbf{R}$ by

$$h(x, y) := d(x_p, y_p) - x_r - y_r - a \quad (x, y \in \Gamma^{**} \boxtimes \mathbf{Z}^{**}).$$

Lemma A.2. *For $x, y, x', y' \in \Gamma^* \boxtimes \mathbf{Z}^*$, it holds $h(x, y) + h(x', y') \geq 2h((x+x')/2, (y+y')/2)$.*

Proof. A classical result [7, Lemma 3] in location theory says that the distance function on a tree is convex. Thus it holds $d(p, q) + d(p', q') \geq 2d((p+p')/2, (q+q')/2)$ for $p, q, p', q' \in \Gamma^*$. The inequality immediately follows from this fact. $\square$

Lemma A.3. *For $x, y \in \Gamma^{**} \boxtimes \mathbf{Z}^{**}$, if $h(x, y) \leq 0$, then $h(\lceil x \rceil, \lceil y \rceil) \leq 0$ and $h(\lfloor x \rfloor, \lfloor y \rfloor) \leq 0$.*

Proof. Let $\Delta := h(\lceil x \rceil, \lceil y \rceil) - h(x, y)$ and $\Delta' := h(\lfloor x \rfloor, \lfloor y \rfloor) - h(x, y)$. Then it holds that $\Delta, \Delta' \in \{-1, -1/2, 0, 1/2, 1\}$. Hence we can assume that $h(x, y) \in \{-1/2, 0\}$. By $a \geq 2$ and $h(x, y) > -1$, it must hold $d(x_p, y_p) > 1$. Then there are two edge-disjoint paths in Γ^{**}, one containing $\lceil x \rceil_p, x_p, \lfloor x \rfloor_p$ and the other containing $\lceil y \rceil_p, y_p, \lfloor y \rfloor_p$. Also they are contained by a single path. From this, we see $d(\lceil x \rceil_p, \lceil y \rceil_p) - d(x_p, y_p) = d(x_p, y_p) - d(\lfloor x \rfloor_p, \lfloor y \rfloor_p)$. Also $\lceil x \rceil_r - x_r = x_r - \lfloor x \rfloor_r$ and $\lceil y \rceil_r - y_r = y_r - \lfloor y \rfloor_r$. Hence we obtain $\Delta' = -\Delta$. Notice that both $h(\lceil x \rceil, \lceil y \rceil)$ and $h(\lfloor x \rfloor, \lfloor y \rfloor)$ are integers. If $\Delta = 1/2$, then $h(x, y) = -1/2$, and hence $h(\lceil x \rceil, \lceil y \rceil) = 0$ and $h(\lfloor x \rfloor, \lfloor y \rfloor) \leq 0$.

So it suffices to show that $h(x, y) = 0$ and $\Delta = 1$ cannot occur. Suppose not. Then both $h(\lceil x \rceil, \lceil y \rceil)$ and $h(\lfloor x \rfloor, \lfloor y \rfloor)$ are odd. If both $\lceil x \rceil$ and $\lceil y \rceil$ (or $\lfloor x \rfloor$ and $\lfloor y \rfloor$) are colored, then they have different colors (Lemma 4.6), $x = \lceil x \rceil = \lfloor x \rfloor$ or $y = \lceil y \rceil = \lfloor y \rfloor$ holds, and $\Delta = 1$ is impossible. Hence we can assume that $x_p, y_p \in \Gamma^{**} \setminus \Gamma^*$, and $(\lceil x \rceil, \lfloor y \rfloor) \in W \times B$ or $(\lfloor x \rfloor, \lceil y \rceil) \in B \times W$. Then $d(\lceil x \rceil_p, \lceil y \rceil_p) - d(x_p, y_p) = d(x_p, y_p) - d(\lfloor x \rfloor_p, \lfloor y \rfloor_p) = 1/2$, $d(\lceil x \rceil_p, \lfloor y \rfloor_p) = d(\lfloor x \rfloor_p, \lceil y \rceil_p) = d(x_p, y_p)$, and $\lceil x \rceil_r - x_r = \lceil y \rceil_r - y_r = x_r - \lfloor x \rfloor_r = y_r - \lfloor y \rfloor_r = -1/4$. Hence $h(\lceil x \rceil, \lfloor y \rfloor) = h(\lfloor x \rfloor, \lceil y \rceil) = h(x, y) = 0$ (even). However $(\lceil x \rceil, \lfloor y \rfloor) \in W \times B$ or $(\lfloor x \rfloor, \lceil y \rceil) \in B \times W$ implies that $h(\lceil x \rceil, \lfloor y \rfloor)$ or $h(\lfloor x \rfloor, \lceil y \rceil)$ is odd; a contradiction. $\square$

We are ready to prove Proposition 3.6. It is not difficult to see that the function on $\Gamma^* \boxtimes \mathbf{Z}^*$ defined by $x \mapsto x_r$ if $x_r \geq 0$ and ∞ otherwise is L-convex. So it suffices to show that the function on $(\Gamma^* \boxtimes \mathbf{Z}^*)^2$ defined by $(x, y) \mapsto 0$ if $h(x, y) \leq 0$ and ∞ otherwise is L-convex. We show that $h(x, y) \leq 0$ and $h(x', y') \leq 0$ imply $h(\lceil (x+x')/2 \rceil, \lceil (y+y')/2 \rceil) \leq 0$ and $h(\lfloor (x+x')/2 \rfloor, \lfloor (y+y')/2 \rfloor) \leq 0$. By Lemma A.2, we have $h((x+x')/2, (y+y')/2) \leq 0$. By Lemma A.3, we have the desired results.